Mean-Variance portfolio selection under partial information

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This talk is based on a joint research with Xun Yu Zhou from Chinese University in Hong Kong.

1. Introduction

n stocks, 1 bond

$$\begin{cases} dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{m} \tilde{\sigma}_{ij}(t)d\tilde{W}_j(t)\right), \\ dS_0(t) = S_0(t)\mu_0(t)dt, \end{cases}$$

$i = 1, 2, \cdots, n$

$$\begin{cases} \mu_i(t) = \text{appreciation rate} \\ \mu_0(t) = \text{interest rate} \\ \tilde{\Sigma}(t) = (\tilde{\sigma}_{ij}(t))_{n\times m} = \text{volatility matrix} \end{cases}$$

Information:

$$\mathcal{G}_t = \sigma(S_i(s) : \ s \le t, \ i = 0, 1, 2, \cdots, n).$$

By Itô's formula,

$$d \log S_i(t) = \left(\mu_i(t) - \frac{1}{2} a_{ii}(t) \right) dt + \sum_{j=1}^{m} \tilde{\sigma}_{ij}(t) d\tilde{W}_j(t)$$

where

$$a_{ij}(t) = \sum_{k=1}^{m} \tilde{\sigma}_{ik}(t) \tilde{\sigma}_{jk}(t).$$

Assumption (ND): $\forall\, t \geq 0$, the $n \times n$-matrix $A(t) \equiv (a_{ij}(t))$ full rank a.s.

Remark: The case $n < m$ (incomplete market) is allowed.

$$Cov(\log S_i, \log S_j)_t = \int_0^t a_{ij}(s)ds.$$

$A(t)$ is $\mathcal{G}_t$-adapted. Defined

$$\Sigma(t) = A(t)^{1/2}$$

Then $\Sigma(t)$ is $\mathcal{G}_t$-adapted.

Martingale representation

$$d\log S_i(t) - \left(\mu_i(t) - \frac{1}{2}a_{ii}(t)\right)dt = \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t) \quad (1)$$

Note

$$\mu_0(t) = \frac{d}{dt} \log S_0(t)$$

is also $\mathcal{G}_t$-adapted.

However, the stochastic process

$$\mu(t) \equiv (\mu_1(t), \cdots, \mu_n(t))$$

is not necessarily $\mathcal{G}_t$-adapted.

Portfolio $u(t) = (u_1(t), \cdots, u_n(t))$.

$$u_i(t) = \text{dollar amount in } i\text{th stock.}$$

Wealth process

$$X(t) = \sum_{i=0}^{n} u_i(t).$$

Remark: The investor bases his decision on available info, i.e., $u(t)$ must be $\mathcal{G}_t$-adapted.

Self-financed

$$dX(t) = \sum_{i=0}^{n} \frac{u_i(t)}{S_i(t)} dS_i(t).$$

Mean-variance selection

Given the mean of the terminal wealth, minimize the variance:

$$\min \left\{ \boldsymbol{Var(X(T))} : \begin{array}{l} \boldsymbol{EX(T) = z,} \\ \boldsymbol{u(t)} \text{ is } \boldsymbol{\mathcal{G}_t}\text{-adapted} \end{array} \right\}.$$

Single period

Markowitz (1952), Merton (1972) Analytic solution

Multi-period

Mossin (1968), Samuelson (1969), ...

Continuous time

Föllmer-Sondermann (1986), Duffie-Jackson (1990),

Duffie-Richardson (1991). Zhou and his coauthors

2. Filtering problem

Suppose that $A(t) = A(\mu(t), \zeta(t))$ where $\zeta(t)$, independent of $\mu(t)$, represents other random factors which may affect $A(t)$, $\mu(t)$ is Markovian w/ generator L.

$$\begin{aligned}\Sigma(t)^{-1} d\log S(t) &- \Sigma(t)^{-1}\left(\mu(t) - \frac{1}{2}diag(A(t))\right) dt \\ &= dW(t)\end{aligned}$$

Denote it as

$$Y(t) = Y(0) + \int_0^t h(s,\mu)ds + W(t).$$

$\mu(t)$ is the signal and $Y(t)$ is the observation. Note that $\mathcal{F}_t^Y = \mathcal{G}_t$. How to estimate $\mu(t)$ based on $\mathcal{G}_t$?

Define measure-valued process

$$\boldsymbol{U}(\boldsymbol{t})(\cdot) = \boldsymbol{E}(\boldsymbol{\mu}(\boldsymbol{t}) \in \cdot|\boldsymbol{\mathcal{G}}_{\boldsymbol{t}}).$$

Namely

$$\langle \boldsymbol{U}(\boldsymbol{t}), \boldsymbol{f} \rangle = \boldsymbol{E}(\boldsymbol{f}(\boldsymbol{\mu}(\boldsymbol{t}))|\boldsymbol{\mathcal{G}}_{\boldsymbol{t}}).$$

How to calculate $\boldsymbol{U}(\boldsymbol{t})$?

Remark: $\boldsymbol{Y}(\boldsymbol{t})$ and $\boldsymbol{\mu}(\boldsymbol{t})$ not independent; $\boldsymbol{W}(\boldsymbol{t})$ and $\boldsymbol{\mu}(\boldsymbol{t})$ independent.

Girsanov formula: Change measure so that $\boldsymbol{Y(t)}$ and $\boldsymbol{\mu(t)}$ independent under new measure.

$$
\begin{aligned}
\frac{d\tilde{P}}{dP} &= M(T) \\
&= \exp\left(\int_0^T h(s,\mu)dY(s) - \frac{1}{2}\int_0^t |h(s,\mu)|^2\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
dM(t) &= M(t)h(t,\mu)dY(t) \\
&= M(t)\left(\mu(t) - \frac{1}{2}diag(A(t))\right)' \Sigma(t)^{-1} dY(t).
\end{aligned}
$$

Under $\tilde{P}$, $\boldsymbol{Y(t)}$ and $\boldsymbol{\mu(t)}$ indep.

Kallianpur-Striebel formula,

$$\boldsymbol{E}\left(\boldsymbol{f}(\boldsymbol{\mu}(\boldsymbol{t}))|\boldsymbol{\mathcal{G}}_{\boldsymbol{t}}\right) = \frac{\langle \boldsymbol{V}(\boldsymbol{t}), \boldsymbol{f}\rangle}{\langle \boldsymbol{V}(\boldsymbol{t}), \mathbf{1}\rangle}$$

where

$$\langle \boldsymbol{V}(\boldsymbol{t}), \boldsymbol{f}\rangle = \tilde{\boldsymbol{E}}\left(\boldsymbol{f}(\boldsymbol{\mu}(\boldsymbol{t}))\boldsymbol{M}(\boldsymbol{t})|\boldsymbol{\mathcal{G}}_{\boldsymbol{t}}\right)$$

is the unnormalized filter.

By the arguments in Kurtz and Xiong (1999), $V(t)$ is represented as

$$\langle V(t), f\rangle = \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} M^i(t) f(\mu^i(t)) \tag{2}$$

where $(\mu^1(t), \zeta^1(t))$, $(\mu^2(t), \zeta^2(t))$, $\cdots$ are independent copies of $(\mu(t), \zeta(t))$ and

$$\begin{aligned} & dM^i(t) \\ = \; & M^i(t)\left(\mu^i(t) - \frac{1}{2} diag(A(\mu^i(t), \zeta^i(t)))\right)' \\ & \quad \Sigma(\mu^i(t), \zeta^i(t))^{-1} dY(t). \end{aligned} \tag{3}$$

Now we derive from Itô's formula and (2-3) a SPDE for the unnormalized filter $V(t)$.

Since $\mu^i(t)$, $i = 1, 2, \cdots$, are independent Markov processes with generator L, there are indepent martingales $N^i(t)$ such that

$$df(\mu^i(t)) = Lf(\mu^i(t))dt + dN^i(t).$$

By Itô's formula, we get

$$d(M^i(t)f(\mu^i(t))) = M^i(t)\left(Lf(\mu^i(t))dt + dN^i(t)\right)$$
$$+f(\mu^i(t))\left(\mu^i(t) - \frac{1}{2}diag A(\mu^i(t), \zeta^i(t))\right)$$
$$\Sigma(\mu^i(t), \zeta^i(t))^{-1}dY(t).$$

Take average for $i = 1, 2, \cdots, k$ and let $k \to \infty$, we see that $V(t)$ satisfies the following Zakai equation:

$$d\langle V(t), f\rangle = \langle V(t), Lf\rangle dt + \langle V(t), G_t f\rangle dY(t) \quad (4)$$

where

$$G_t f(\mu) = f(\mu) E\Big\{ \Big(\mu - \frac{1}{2} diag A(\mu, \zeta(t)) \Big) \Sigma(\mu, \zeta(t))^{-1} \Big\}.$$

Apply Kallianpur-Striebel and Itô, we get

$$\begin{aligned} & d \langle U(t), f \rangle \\ = \; & \langle U(t), Lf \rangle \, dt \\ & + (\langle U(t), G_t f \rangle - \langle U(t), f \rangle \langle U(t), G_t 1 \rangle) \, d\nu(t) \end{aligned}$$

where

$$\nu(t) = Y(t) - \int_0^t \langle U(s), G_s 1 \rangle \, ds$$

is the innovation process.

3. Optimization

$$\begin{aligned} dX(t) &= \sum_{i=0}^{n} \frac{u_i(t)}{S_i(t)} dS_i(t) \\ &= \left(X(t)\mu_0(t) + \sum_{i=1}^{n} (\mu_i(t) - \mu_0(t)) u_i(t) \right) dt \\ &\quad + \sum_{i,j=1}^{n} \sigma_{ij}(t) u_i(t) dW_j(t). \end{aligned}$$

Note that $\mu_i(t)$ and $W_j(t)$ are not $\mathcal{G}_t$-adapted.

Take expectation

$$
\begin{aligned}
\boldsymbol{X}(t) &= \boldsymbol{E}(\boldsymbol{X}(t)|\mathcal{G}_t) \\
&= x_0 + \sum_{i,j=1}^{n} \int_0^t \sigma_{ij}(s) u_i(t) d\nu_j(s) \\
&\quad + \int_0^t \left(X(s)\mu_0(s) + \sum_{i=1}^{n} (\bar{\mu}_i(s) - \mu_0(s)) u_i(s) \right) ds
\end{aligned}
$$

where innovation process $\nu(t)$ is defined by

$$
\begin{aligned}
\nu(t) &\equiv E(W(t)|\mathcal{G}_t) \\
&= Y(t) - \int_0^t \langle U(s), G_s 1 \rangle \, ds.
\end{aligned}
$$

Then $\nu(t)$ is martingale with

$$\langle \nu \rangle_t = tI.$$

So it is a $\mathcal{G}_t$-B.M.

Let $\bar{\mu}_i(t) = E(\mu_i(t)|\mathcal{G}_t)$. Then

$$W(t) = \nu(t) + \int_0^t \Sigma(s)^{-1}(\bar{\mu}(s) - \mu(s))ds.$$

Hence

$$\begin{aligned} dX(t) &= \left(X(t)\mu_0(t) + \sum_{i=1}^n (\bar{\mu}_i(t) - \mu_0(t))u_i(t) \right) dt \\ &\quad + \sum_{i,j=1}^n \sigma_{ij}(t)u_i(t)d\nu_j(t). \end{aligned} \tag{5}$$

The optimization problem becomes

$$\min\left(E(X(T)-z)^2\right)$$

subject to

$$EX(T)=z, \qquad (5), \qquad u_i(t) \text{ is } \mathcal{G}_t\text{-adapted.}$$

Define

$$b_j(t)=\sum_{i=1}^{n}\sigma_{ij}^{-1}(t)(\bar{\mu}_i(t)-\mu_0(t))$$

and

$$d\rho(t)=-\rho(t)\mu_0(t)dt-\sum_{j=1}^{n}b_j(t)\rho(t)d\nu_j(t). \tag{6}$$

Applying Itô's formula to (6) and (5), we have

$$X(t) = \rho(t)^{-1} E(\rho(T)X(T)|\mathcal{G}_t). \tag{7}$$

Take $t = 0$, we have

$$E(\rho(T)X(T)) = x_0. \tag{8}$$

We derive the optimal strategy in two steps. First, we solve a static optimization problem subject to constraint (8) to find the best $X^*(T)$. Finally, we show that there is a portfolio such that $X^*(T)$ is its terminal wealth.

(a) Static problem

$$\min\left(E(X(T)-z)^2\right)$$

subject to

$$EX(T)=z, \qquad E(\rho(T)x(T))=x_0.$$

Then

$$\begin{aligned} X(T) &= z+\frac{x_0-zE\rho(T)}{Var(\rho(T))}(\rho(T)-E\rho(T)) \\ &\equiv v. \end{aligned} \tag{9}$$

(b) Replicate $\boldsymbol{v}$

We seek the wealth process $\boldsymbol{X(t)}$ which satisfies (5) and $\boldsymbol{X(T) = v}$. Namely, we seek a solution to the following BSDE:

$$\begin{cases} dX(t) = \Big(X(t)\mu_0(t) + \sum_{j=1}^{n}(\bar{\mu}_j(t) - \mu_0(t))u_j(t)\Big)\,dt \\ \qquad + \sum_{i,j=1}^{n} \sigma_{ij}(t)u_i(t)d\nu_j(t) \\ X(T) = v. \end{cases} \tag{10}$$

Let

$$Z_j(t) = \sum_{i=1}^{n} \sigma_{ij}(t)u_i(t).$$

Then

$$\boldsymbol{u_i(t) = \sum_{j=1}^{n} \sigma_{ij}^{-1}(t) Z_j(t).}$$

Then, (10) becomes

$$\begin{cases} \boldsymbol{dX(t) = \left(X(t)\mu_0(t) + \sum_{j=1}^{n} b_j(t) Z_j(t) \right) dt} \\ \qquad \boldsymbol{+ \sum_{j=1}^{n} Z_j(t) d\nu_j(t)} \\ \boldsymbol{X(T) = v.} \end{cases} \tag{11}$$

If $\boldsymbol{\mathcal{F}_t^{\nu} = \mathcal{G}_t}$, (11) is the usual BSDE. However, it is well-known that, in general, $\boldsymbol{\mathcal{F}_t^{\nu} \neq \mathcal{G}_t}$.

Thm: (11) has a solution. Further, the optimal portfolio is given by

$$
\begin{aligned}
u_i(t) &= \sum_{j=1}^{n} a_{ij}^{-1}(t)(\bar{\mu}_j(t) - \mu_0(t)) \\
&\quad + \rho(t)^{-1} \sum_{j=1}^{n} \sigma_{ij}^{-1}(t)\eta_j(t)
\end{aligned}
$$

where

$$\eta_j(t) = \frac{d}{dt}\langle N, \nu_j \rangle_t$$

and

$$N(t) = E(\rho(T)v|\mathcal{G}_t).$$

Numerical sol. for optimal portfolio

recall (6)

$$d\rho(t) = -\rho(t)\mu_0(t)dt - \sum_{j=1}^{n} b_j(t)\rho(t)d\nu_j(t).$$

Let $(b^1, \nu^1),\ (b^2, \nu^2),\ \cdots$ be indep. copies of (b, ν).

Define

$$\rho^i(t, t') = \rho(t) \qquad t \leq t'$$

For $t > t'$, let $\rho^i(t, t')$ be defined by (6) with (b, ν) replaced by

(b^i, ν^i). Then

$$N(t) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \rho^i(T,t) v^i(T,t)$$

where

$$v^i(T,t) = z + \frac{x_0 - zE\rho^i(T,t)}{Var(\rho^i(T,t))}(\rho^i(T,t) - E\rho^i(T,t)).$$