Several types of uniqueness of Kolmogorov forward equation and semogroup of kernels

Liming Wu

Wuhan University, China
Blaise Pascal University, France

Beijing Normal University
10-14 Augest 2004

1 Introduction

(I) Two ways of ragarding the uniqueness

Given a space of functions $\mathbb{B}$ on some Polish space $(E, \mathcal{B})$ and a pre-generator $\mathcal{L}$ acting on some test-functions space $\mathcal{D} \subset \mathbb{B}$.

1) Mathematical way: Whether is-there a unique "semigroup" (P_t) on $\mathbb{B}$ such that its generator $\hat{\mathcal{L}}$ extends $\mathcal{L}$?

2) Physical way: the Kolmogorov forward equation below

$$\partial_t \nu_t = \mathcal{L}^* \nu_t, \quad \nu_0 \text{ given} \qquad \text{(KFE)}$$

is-it well-posed ?

Definite answer in the framework of C_0-semigroup:

Theorem 1.1. *Let $\mathbb{B}$ be a Banach space and (P_t) a C_0-semigroup on $\mathbb{B}$ such that its generator $\hat{\mathcal{L}}$ extends $(\mathcal{L}, \mathcal{D})$, where $\mathcal{D}$ is dense in ~~$\mathcal{B}$.~~ $\mathbb{B}$
The following properties are equivalent:*

(i) (P_t) is the unique C_0-semigroup on $\mathbb{B}$ such that its generator $\hat{\mathcal{L}}$ extends $(\mathcal{L}, \mathcal{D})$;

(ii) $\mathcal{D}$ is a core for $\hat{\mathcal{L}}$, i.e. $\bar{\mathcal{L}} = \hat{\mathcal{L}}$;

(iii) (Liouville property) for some or all large enough $\lambda \in \mathbb{R}$, $Ker(\lambda - \mathcal{L}^*) = \{0\}$;

(iv) KFE is well-posed or has a unique solution in $\mathbb{B}^*$;

(v) the abstract Cauchy problem

$$\partial_t u_t = \bar{\mathcal{L}} u_t, \quad u_0 \text{ given}$$

is well-posed. ↳ closure of $\mathcal{L}$

Called : "$\mathbb{B}$-uniqueness"

Remarks 1.2. *This beautiful result can be applied in*

$$\mathbb{B} = L^p(\mu),\ 1 \le p < +\infty;\ \ \mathbb{B} = \bar{C}_0(E),\ E \text{ locally compact};$$

but CAN NOT be applied in the following situations:

1) $\mathbb{B} = L^\infty(\mu)$; *(Y.P. Zhang and Wu 01, CRAS)*

2) $\mathbb{B} = C_b(E)$ *where* E *is infinite-dimensional (Y.P. Zhang and Wu 01-04, in preparation);*

3) $\mathbb{B} = b\mathcal{B}$ *(this work).*

Indeed Lotz (1983) and T. Coulhon (1984) proved that if (P_t) *is a* C_0*-semigroup on* L^∞ *or* $b\mathcal{B}$*, then its generator* $\hat{\mathcal{L}}$ *is bounded.*

Remarks 1.3. *Four sentences:*

1. *for a unbounded operator* $\hat{\mathcal{L}}$*, knowing its full domain* $\mathbb{D}(\hat{\mathcal{L}})$ *is very important;*

2. *it is very difficult (to do not say that is impossible usually) to describe* $\mathbb{D}(\hat{\mathcal{L}})$*;*

3. *hence to determine if a test-functions space* $\mathcal{D}$ *is a core for* $\hat{\mathcal{L}}$ *is almost the only fashion to know* $\hat{\mathcal{L}}$*;*

4. *the problem of core is equivalenet to the problem of the uniqueness.*

This talk treats the semigroups of kernels on $\mathbb{B} = b\mathcal{B}$.

Example 1:

$K \subset \mathbb{R}^n$ closed, $1 < p < +\infty$

- $(\Delta, C_0^\infty(\mathbb{R}^n \setminus K))$ is $L^p(dx)$-unique iff $Cap_{2,p}(K) = 0$
- $(\Delta, C_0^\infty(\mathbb{R}^n \setminus K))$ is $L^1(dx)$-unique iff it is Markov unique in the Dirichlet form sense

iff $Cap_{1,2}(K) = 0$

iff the BM does not hit K!

Example 2. M complete connected Riemannian manifold without boundary

- $(\Delta_M, C_0^\infty(M))$ is $L^p(dx)$-unique for every $p \in (1, +\infty)$ [Yau-Strichartz)
- It is $L^1(dx)$-unique iff M is stoch. complete (the BM does not explodes). [Davies]
- It is $L^\infty(dx)$-unique if

$$Ric(x) \geq -c(1 + \rho(x, o)^2)$$

Example 3. potential $V \geq 0$ locally in L^1.

$\left(-\frac{1}{2}\Delta + V,\ C_0^\infty(D)\right)$ is $L^1_{D}(dx)$-unique iff

$$\mathbb{P}_x\left(\int_0^{\sigma_D} (1 + V(B_s))\, ds = +\infty\right) = 1, \quad dx\text{-a.e}$$

$\left(-\frac{1}{2}\Delta + V,\ C_0^\infty(D)\right)$ is Dirichlet-Markov unique iff

$$\mathbb{P}_x\left(\int_0^{\sigma_D + \varepsilon} (1 + V(B_s))\, ds = +\infty,\ \forall \varepsilon > 0\right) = 1$$

dx-a.e

(W 98, JFA)

2 Preliminaries on semigroups of kernels

Two classical books:

Dynkin: Markov processes, Vol. I and II (1963, 64)

Ethier-Kurtz Markov processes: characterization and convergence (1986)

2.1 Notations and definition

$b\mathcal{B}$: the Banach space of all real and bounded $\mathcal{B}$-measurable functions on E, equipped with the sup norm $\|f\| := \sup_{x\in E} |f(x)|$; and

$M_b(E)$: the space of all (maybe signed) σ-additive measures ν such that its variation $\|\nu\|_{var} < +\infty$.

Then

$$\langle \nu, f\rangle := \nu(f) := \int_E f(x)\nu(dx),\ \forall(\nu, f) \in M_b(E) \times b\mathcal{B}$$

is a bilinear form of duality between $b\mathcal{B}$ and $M_b(E)$.

Definition 2.1. *A family $(P_t)_{t\in\mathbb{R}^+}$ of bounded operators on $b\mathcal{B}$ is said to be a semigroup of bounded operators on $b\mathcal{B}$, denoted by $(P_t) \in \mathcal{SG}$, if*

(i) $P_0 = Id$ (the identity operator, it can be represented by the kernel $P_0(x, \cdot) = \delta_x$ (the Dirac measure at x), for all $x \in E$);

(ii) for all $s, t \in \mathbb{R}^+$, $P_sP_t = P_{s+t}$;

(iii) for each $f \in b\mathcal{B}$, $(t, x) \to P_tf(x)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}$-measurable on $\mathbb{R}^+ \times E$;

(iv) there are $C > 0$ and $\kappa \in \mathbb{R}$ such that $\|P_t\| \le Ce^{\kappa t}, \forall t \in \mathbb{R}^+$.

If the constants C, κ are given, we write $(P_t) \in \mathcal{SG}(C, \kappa)$.

If moreover every P_t can be represented by a bounded kernel $P_t(x, dy)$, we say that $(P_t)_{t\in\mathbb{R}^+}$ is a semigroup of bounded kernels, denoted by $(P_t) \in \mathcal{SG}_K$. Similarly we write $(P_t) \in \mathcal{SG}_K(C, \kappa)$ if and only if $(P_t) \in \mathcal{SG}(C, \kappa)$ and $(P_t) \in \mathcal{SG}_K$.

Given $(P_t) \in \mathcal{SG}_K(C, \kappa)$, if $P_t(x, \cdot)$ ($t \ge 0$, $x \in E$) are nonnegative measures (i.e., nonnegative kernels), we write $(P_t) \in \mathcal{SG}_K^+(C, \kappa)$. In *particular, $(P_t) \in \mathcal{SG}_K^+(1,0)$ iff it is a semigroup of sub-Markov kernels verifying the measurable condition (iii).*

Given $(P_t) \in \mathcal{SG}_K(C,\kappa)$, then for any $\lambda > \kappa$,

$$R_\lambda f(x) := \int_0^\infty e^{-\lambda t} P_t f(x) dt, \ \forall f \in b\mathcal{B} \tag{2.1}$$

defines not only a bounded linear operator on $b\mathcal{B}$ (the resolvent), but also a bounded kernel.

To see the difference between bounded operators on $b\mathcal{B}$ and bounded kernels, let us consider

Definition 2.2. *On $b\mathcal{B}$, we denote by σ_m the weak topology $\sigma(b\mathcal{B}, M_b(E))$, i.e., the weakest topology on $b\mathcal{B}$ such that all linear forms $f \to \langle \nu, f\rangle$ where $\nu \in M_b(E)$ are continuous.*

Lemma 2.3. *Let $P : b\mathcal{B} \to b\mathcal{B}$ be a bounded linear operator. Then the following properties are equivalent:*

(i) P is continuous on $(b\mathcal{B}, \sigma_m)$;

(ii) P can be identified as a kernel, i.e., there is a bounded kernel $P(x, dy)$ such that

$$Pf(x) = \int_E f(y) P(x, dy);$$

(iii) If $f_n \to 0$ in the bp-convergence, then $Pf_n(x) \to 0$ for all $x \in E$.

Lemma 2.4. *Let $\mathcal{D}$ be a linear subspace of $b\mathcal{B}$. Then $\mathcal{D}$ is dense in $(b\mathcal{B}, \sigma_m)$ iff for any given $\nu \in M_b(E)$,*

$$(\langle \nu, f\rangle = 0, \ \forall f \in \mathcal{D}) \implies \nu = 0$$

i.e., $\mathcal{D}$ is separating on E in the language of Ethier-Kurtz [4].

Proof. Since $(b\mathcal{B}, \sigma_m)' = M_b(E)$ and σ_m is a locally convex topology, this characterization is just an immediate consequence of the Hahn-Banach theorem. □

2.2 Multi-valued operator and the full generator

A sub-linear space A of $b\mathcal{B} \times b\mathcal{B}$ will be called a multi-valued linear operator (or the graph of a multi-valued linear operator), with domain of definition $\mathbb{D}(A) := \{f/\exists g \in b\mathcal{B} \text{ such that } (f,g) \in A\}$ and range $Ran(A) := \{g/\exists f \text{ such that } (f,g) \in A\}$. For $f \in \mathbb{D}(A)$, $Af := \{g/(f,g) \in A\}$. If Af is a singleton $\{g\}$, we write simply $g = Af$. Finally let

$$\lambda - A := \{(f, \lambda f - g)/(f,g) \in A\}, \ A^{-1} := \{(g,f)/ \ (f,g) \in A\}.$$

We define now the full generator $\hat{\mathcal{L}}$ of a semigroup (P_t).

Definition 2.5. *Given $(P_t) \in \mathcal{SG}$. Its full generator $\hat{\mathcal{L}}$ is the linear subspace in $b\mathcal{B} \times b\mathcal{B}$ characterized by $(f,g) \in \hat{\mathcal{L}}$ iff*

$$P_t f(x) - f(x) = \int_0^t P_s g(x) ds, \ \forall t \geq 0, \ x \in E. \tag{2.2}$$

Example 2.6. Let $(P_t(x,dy))$ be the transition semigroup of the Brownian Motion on $E = \mathbb{R}^d$ with (formal) generator $\Delta/2$. Then its full generator $\hat{\mathcal{L}}$ can be characterized as

(i) Given $f \in b\mathcal{B}(\mathbb{R}^d)$, $f \in \mathbb{D}(\hat{\mathcal{L}})$ iff $f \in C_b(E)$, the space of real bounded and continuous functions on $E = \mathbb{R}^d$, and $\Delta f \in L^\infty(\mathbb{R}^d, dx)$ in the sense of distribution of Schwartz;

(ii) Given $f \in \mathbb{D}(\hat{\mathcal{L}})$ and $g \in b\mathcal{B}(\mathbb{R}^d)$, then $g \in \hat{\mathcal{L}} f$ iff $g = \Delta f/2$, $dx-$ a.e.

(we remind the reader that this is quite difficult.)

However for more complicated diffusion semigroups (especially those generated by a degenerate elliptic second-order differential operator), it is very difficult to describe exactly $\hat{\mathcal{L}}$.

Notice that if $f \in \mathbb{D}(\hat{\mathcal{L}})$, then $\lim_{t\to 0} \|P_t f - f\| = 0$, i.e., f belongs to

$$\mathbb{B}_u := \{h \in b\mathcal{B}/ \lim_{t\to 0} \|P_t h - h\| = 0\} \tag{2.3}$$

on which (P_t) is stable and becomes a C_0-semigroup.

Lemma 2.7. *Given $(P_t) \in \mathcal{SG}$. Let $\mathcal{L}_u$ be the uniform generator of (P_t), i.e., the generator of $(P_t|_{\mathbb{B}_u})$. Then*

$$\mathcal{L}_u = \{(f,g) \in b\mathcal{B} \times b\mathcal{B};\ \lim_{t\to 0} \left\| \frac{P_t f - f}{t} - g \right\| = 0\}$$
$$= \hat{\mathcal{L}} \bigcap (\overline{\mathbb{D}(\hat{\mathcal{L}})} \times \overline{\mathbb{D}(\hat{\mathcal{L}})})$$

and $\mathbb{B}_u = \overline{\mathbb{D}(\mathcal{L}_u)} = \overline{\mathbb{D}(\hat{\mathcal{L}})}$.

Theorem 2.8. *(Dynkin [2] (65)) Given two semigroups of bounded operators in $\mathcal{SG}$, if they have the same uniform generator $\mathcal{L}_u$, then $\tilde{P}_t|_{\mathbb{B}_u} = P_t|_{\mathbb{B}_u}$. In particular $\tilde{P}_t|_{\mathbb{B}_u} = P_t|_{\mathbb{B}_u}$ if they have the same full generator $\hat{\mathcal{L}}$.*

Proposition 2.9. *Let $\hat{\mathcal{L}}$ be the full generator of $(P_t) \in \mathcal{SG}_K(C,\kappa)$.*
The following properties are equivalent:

(i) $\mathbb{D}(\hat{\mathcal{L}})$ *is dense in* $(b\mathcal{B}, \sigma_m)$*;*

(ii) the strongly continuous subspace $\mathbb{B}_u$ given in (2.3) is dense in $(b\mathcal{B}, \sigma_m)$;

(iii) the pointwise continuous subspace

$$\mathbb{B}_{\sigma_c} := \{f \in b\mathcal{B};\ P_t f(x) \ \ \text{is continuous on } \mathbb{R}^+,\ \forall x \in E\}$$

is dense in $(b\mathcal{B}, \sigma_m)$*;*

(c.iv)

$$\mathbb{B}_\sigma = \{f \in b\mathcal{B};\ \lim_{t\to 0+} P_t f(x) = f(x),\ \forall x \in E\}$$

is dense in $(b\mathcal{B}, \sigma_m)$.

Moreover $P_t(\mathbb{B}_{\sigma_c}) \subset \mathbb{B}_{\sigma_c}$ *and* $P_t(\mathbb{B}_\sigma) \subset \mathbb{B}_\sigma$.

The equivalences above motivates

Definition 2.10. Given $(P_t) \in \mathcal{SG}_K$, if $\mathbb{B}_\sigma$ is dense in $(b\mathcal{B}, \sigma_m)$ or equivalently if it is separating on E, we say that (P_t) is a *regular* semigroup of kernels, denoted by $(P_t) \in \mathcal{SG}_K^r$.

If $\mathbb{B}_\sigma = b\mathcal{B}$, we say that (P_t) is *completely regular.*

Lemma 2.11. *Given $(P_t) \in \mathcal{SG}_K$, if (P_t) verifies: $P_t(x,\cdot) \to \delta_x$ in the weak convergence of measures as $t \to 0+$, then (P_t) is regular.*

Proof. By the assumption, $C_b(E) \subset \mathbb{B}_\sigma$. □

In the framework of sub-Markov semigroup, the assumption in the lemma above is the so called *stochastic continuity* (cf. Dynkin [2]).

The most often encountered (sub)Markov semigroups are regular, but not completely regular;

for a sub-Markov semigroup $(P_t) \in \mathcal{SG}_K^+(0,1)$, if it is completely regular, then it is of pur jump type by M.F. Chen [1].

Open Question: *prove (or disprove) that all $(P_t) \in \mathcal{SG}_K$ are regular.*

Theorem 2.12. *Let $(P_t), (\tilde{P}_t) \in \mathcal{SG}_K^r$. If they have the same full generator $\hat{\mathcal{L}}$ or the same uniform generator $\mathcal{L}_u$, then $P_t = \tilde{P}_t$ over $b\mathcal{B}$.*

2.3 Uniqueness of the abstract Cauchy problem

Theorem 2.13. *Let $\hat{\mathcal{L}}$ be the full generator of $(P_t) \in \mathcal{SG}_K(C,\kappa)$. If $u(t)$ is a solution of*

$$\frac{d}{dt}u(t) \in \hat{\mathcal{L}}u(t), \ u(0) = f \in \mathbb{D}(\hat{\mathcal{L}}) \ (given) \tag{2.4}$$

in the following sense:

(i) $u(t) \in \mathbb{D}(\hat{\mathcal{L}})$ and there is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}$-measurable function $g(t,x)$ such that $g(t,\cdot) \in \hat{\mathcal{L}}u(t)$ and $\sup_{s \le t} \|g(s,\cdot)\| < +\infty$ for all $t \ge 0$; and

(ii)

$$u(t,x) - f(x) = \int_0^t g(s,x)ds, \ \forall (t,x) \in \mathbb{R}^+ \times E;$$

and if moreover

$$\|u(t)\| \le C_1 e^{C_2 t}, \ \forall t \ge 0 \tag{2.5}$$

for some constants $C_1, C_2 \ge 0$, then $u(t,x) = P_t f(x)$.

3 Uniqueness of Kolmogorov's equation and of regular semigroup of kernels

3.1 Uniqueness of Kolmogorov's forward equation

Given a single valued operator $\mathcal{L}$ acting on a space of test-functions $\mathcal{D} \subset b\mathcal{B}$.

Definition 3.1. *A flow $t \to \nu_t \in M_b(E)$ indexed by $t \in \mathbb{R}^+$ is said to be a measure-valued weak solution of the Kolmogorov forward equation*

$$\frac{d}{dt}\nu_t = \mathcal{L}^*\nu_t \tag{3.1}$$

with initial condition $\nu_0 = \nu$, if

(i) $t \to \nu_t(g)$ is $\mathcal{B}(\mathbb{R}^+)$-measurable for all $g \in b\mathcal{B}$;

ii $\int_0^t |\langle \nu_s, \mathcal{L}f\rangle| ds < +\infty$ for all $t \geq 0$; and

(iii) for all $t \geq 0$ and all $f \in \mathcal{D}$,

$$\langle \nu_t, f\rangle - \langle \nu, f\rangle = \int_0^t \langle \nu_s, \mathcal{L}f\rangle ds \tag{3.2}$$

A first main new result of this work is

Theorem 3.2. *Assume that $(P_t) \in \mathcal{SG}_K(C, \kappa)$ with the full generator $\hat{\mathcal{L}}$. Given a single valued linear operator $\mathcal{L}$ on $b\mathcal{B}$ with domain $\mathcal{D}$, such that $\mathcal{L} \subset \hat{\mathcal{L}}$. Then*

(a) The following properties are equivalent.

(i) $\mathcal{D}$ is a core for $\hat{\mathcal{L}}$ w.r.t. the σ_m-topology, i.e., identifying $\mathcal{L}$ as its graph in $b\mathcal{B} \times b\mathcal{B}$, we have

$$\overline{\mathcal{L}}^{\sigma_m} = \hat{\mathcal{L}}.$$

(ii) $\mathcal{L}^* = \hat{\mathcal{L}}^*$.

(iii) For all $\lambda > \kappa$, $(\lambda - \mathcal{L})(\mathcal{D})$ is dense in $(b\mathcal{B}, \sigma_m)$, or equivalently for any given $\nu \in M_b(E)$,

$$(\langle \nu, (\lambda - \mathcal{L})f\rangle = 0,\ \forall f \in \mathcal{D}) \implies \nu = 0. \tag{3.3}$$

(iv) *Property (iii) holds for some* $\lambda > \kappa$, *i.e, (3.3) holds for some* $\lambda > \kappa$.

(b) *Assume moreover that* $\mathcal{D}$ *is dense in* $(b\mathcal{B}, \sigma_m)$ *(hence* $\mathbb{B}_{\sigma_c} \supset \mathcal{D}$ *is separating, i.e.,* (P_t) *is regular). Then each of the equivalent conditions in part (a) is equivalent to*

(v) *Given any initial measure* $\nu \in M_b(E)$, *the Kolmogorov forward equation (3.1) associated* $(\mathcal{L}, \mathcal{D})$ *has a unique weak measure-valued solution* (ν_t) *with* $\nu_0 = \nu$ *so that there are constants* $C_1, C_2 > 0$,

$$\|\nu_t\|_{var} \le C_1 e^{C_2 t}, \forall t \ge 0. \tag{3.4}$$

Moreover this solution is given by $\nu_t = \nu P_t$.

Definition 3.3. *Given a single valued linear operator* $\mathcal{L}$ *with* $\mathbb{D}(\mathcal{L}) = \mathcal{D}$ *on* $b\mathcal{B}$,

(a) *we say that* $\mathcal{L}$ *is Kolmogorov-forward (KF in short) unique, if for each* $\nu \in M_b(E)$, *the Kolmogorov forward equation (3.1) has at most one solution* (ν_t) *verifying moreover* $\nu_t \in M_b(E)$ *and (3.4) (i.e., the uniqueness in Theorem 3.2(v) holds);*

(b) *we say that* $\mathcal{L}$ *is KF*$^+$*-unique, if for each* $\nu \in M_b^+(E)$, *the Kolmogorov forward equation (3.1) has at most one solution* (ν_t) *verifying moreover* $\nu_t \in M_b^+(E)$ *and (3.4);*

(c) *we say that* $\mathcal{L}$ *is KFM-unique ("M" means sub-Markov), if for each* $\nu \in M_1(E)$, *the Kolmogorov forward equation (3.1) has at most one solution* (ν_t) *verifying moreover* $\nu_t \in M_{\le 1}(E)$.

Obviously

KF uniqueness $\Longrightarrow$ KF$^+$-uniqueness $\Longrightarrow$ KFM uniqueness.

For the heat diffusion phenomena described by the Kolmogorov forward (or Fokker-Planck) equation (3.1), where ν_t denotes the temperature distribution at time t, the KF$^+$-uniqueness is very natural for the temperature is always nonnegative. When the system has no outer positive heat source, the total heat $\nu_t(E)$ at time t will not bypass the total heat $\nu_0(E) = \nu(E)$ at time 0, and so the KFM-uniqueness is also natural in the actual situation.

3.2 Uniqueness of semigroup of kernels

Parallel to Definition 3.3, we give

Definition 3.4. *Given a single valued linear operator $\mathcal{L}$ with $\mathbb{D}(\mathcal{L}) = \mathcal{D}$ on $b\mathcal{B}$,*

(a) we say that $\mathcal{L}$ is $b\mathcal{B}$-unique, if there is at most one semigroup $(P_t) \in \mathcal{SG}_K$ such that its full generator $\hat{\mathcal{L}} \supset \mathcal{L}$;

(b) we say that $\mathcal{L}$ is $b\mathcal{B}^+$-unique, if there is at most one nonnegative semigroup $(P_t) \in \mathcal{SG}_K^+$ such that its full generator $\hat{\mathcal{L}} \supset \mathcal{L}$;

(c) we say that $\mathcal{L}$ is Markov-unique, if there is at most one sub-Markov semigroup $(P_t) \in \mathcal{SG}_K^+(1,0)$ such that its full generator $\hat{\mathcal{L}} \supset \mathcal{L}$;

We have obviously

Proposition 3.5. *Given a single valued linear operator $\mathcal{L}$ on $b\mathcal{B}$, with $\mathbb{D}(\mathcal{L}) = \mathcal{D}$ dense in $(b\mathcal{B}, \sigma_m)$.*

(a) If $(\mathcal{L}, \mathcal{D})$ is KF-unique, then it is $b\mathcal{B}$-unique.

(b) If $(\mathcal{L}, \mathcal{D})$ is KF$^+$-unique, then it is $b\mathcal{B}^+$-unique.

(c) If $(\mathcal{L}, \mathcal{D})$ is KFM-unique, then it is Markov-unique.

I believe that the inverses of all the three implications above are true.

3.3 Several corollaries

3.3.1 A result of Ethier-Kurtz revisited

Corollary 3.6. *Assume that $(P_t) \in \mathcal{SG}_K(C, \kappa)$ with the full generator $\hat{\mathcal{L}}$. Given a single valued linear operator $\mathcal{L}$ on $b\mathcal{B}$ with domain $\mathcal{D}$ separating on E, such that $\mathcal{L} \subset \mathcal{L}_u$. If for some $\lambda > \kappa$,*

$$\overline{(\lambda - \mathcal{L})(\mathcal{D})}^{\|\cdot\|} \supset \mathcal{D}, \tag{3.5}$$

then all conclusions in Theorem 3.2 hold. If moreover $\overline{\mathcal{D}}^{\|\cdot\|} = \mathbb{B}_u$, then $\overline{\mathcal{L}}^{\|\cdot\|} = \mathcal{L}_u$.

Let us see a baby model.

Example 3.7. Let $V \geq 0$ be a real measurable function on E. Consider the semigroup $P_t f(x) := e^{-tV(x)} f(x)$ or $P_t(x, dy) = e^{-tV(x)}\delta_x(dy)$, i.e., $(P_t) \in \mathcal{SG}_K^+(1, 0)$. Its full generator is single valued given by

$$\mathbb{D}(\hat{\mathcal{L}}) = b_V\mathcal{B} := \{f \in b\mathcal{B};\, \|f\|_V := \sup_x |f(x)|(1 + V(x))^{-1} < +\infty\},$$

$$\hat{\mathcal{L}} f = -V f,\ \forall f \in b_V\mathcal{B}.$$

and

$$\mathbb{B}_u = \overline{\mathbb{D}(\hat{\mathcal{L}})} = \{f \in b\mathcal{B};\ \lim_{n\to\infty} \sup_{x\in[V>n]} |f(x)| = 0\}.$$

For this example we see that $\mathbb{B}_{\sigma_c} = \mathbb{B}_\sigma = b\mathcal{B}$ and $\mathcal{L}_{\sigma_c} = \mathcal{L}_\sigma = \mathcal{L}_{bp} = \hat{\mathcal{L}}$.

Assume that V is unbounded and consider the space $\mathcal{D}$ of test-functions $f \in b\mathcal{B}$ such that $[f \neq 0] \subset E_n := [V \leq n]$ for some $n \in \mathbb{N}\}$. Let $\mathcal{L} := \hat{\mathcal{L}}|_{\mathcal{D}}$.

Since $(\lambda - \mathcal{L})\mathcal{D} = \{(\lambda + V)f;\ f \in \mathcal{D}\} = \mathcal{D}$ we see that $\mathcal{L}$ verifies (3.5). Hence (P_t) is the only semigroup of kernels generated by $\mathcal{L}$, and the Kolmogorov forward equation (3.1) has a unique condition verifying (3.4). Moreover since $\overline{\mathcal{D}} = b_V\mathcal{B} = \mathbb{B}_u$, we also have $\overline{\mathcal{L}}^{\|\cdot\|} = \mathcal{L}_u$.

But for this simple example, $\mathcal{L}$ is Q-unique (Hou and Chen) iff V is bounded.

4 Martingale uniqueness and Markov uniqueness

Given a single-valued linear operator $\mathcal{L} : \mathcal{D} \to b\mathcal{B}$ with domain $\mathcal{D}$ which is a linear subspace of $b\mathcal{B}$. Let ν be an element in the space $M_1(E)$ of probability measures on E.

Definition 4.1. *By a martingale solution associated with* $(\mathcal{L}, \mathcal{D}, \nu)$, *we mean a stochastic process* $(X_t)_{t\geq 0}$ *valued in some larger measurable space* $(\bar{E}, \bar{\mathcal{B}})$ *such that* $\bar{E} \supset E$, $\bar{\mathcal{B}} \bigcap E := \{A \cap E;\ A \in \bar{E}\} \supset \mathcal{B}$, *defined on some complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

(i) $\mathbb{P}(X_0 \in \cdot \cap E) = \nu(\cdot)$;

(ii) *For each* $t \geq 0$, $[X_t \in E] \in \mathcal{F}$ *and* $(t, \omega) \to 1_E(X_t(\omega))$ *is measurable on* $(\mathbb{R}^+ \times \Omega, \mathcal{A})$ *where* $\mathcal{A}$ *is the completion of* $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$ *is by* $dt \otimes \mathbb{P}$;

(iii) For every $f \in \mathcal{D}$, $M_t(f) := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ (which is well defined by (ii)) is a martingale w.r.t. the filtration

$$\mathcal{G}_t := \text{completion of } \sigma(X_s, \int_0^s g(X_u)du;\ s \le t, g \in b\bar{\mathcal{B}}) \text{ by } \mathbb{P}.$$

Here, a real function f on E is interpreted as a function on $\bar{E}$ with the convention that $f|_{\bar{E}\setminus E} = 0$.

Definition 4.2. *We say that the solution of the martingale problem associated to $(\mathcal{L}, \mathcal{D}, \nu)$ is unique, if for any two martingale solutions (X_t) and (Y_t) associated with $(\mathcal{L}, \mathcal{D}, \nu)$ have the same finite-dimensional distributions on E, i.e.,*

$$\mathbb{P}(X_{t_1} \in A_1, \cdots, X_{t_n} \in A_n) = \mathbb{P}(Y_{t_1} \in A_1, \cdots, Y_{t_n} \in A_n)$$

for all $0 \le t_1 < t_1 < \cdots < t_n$ and $A_i \in \mathcal{B}, i = 1, \cdots, n$.

We say that $(\mathcal{L}, \mathcal{D})$ is martingale-unique, if the solution of the martingale problem associated to $(\mathcal{L}, \mathcal{D}, \nu)$ is unique for each $\nu \in M_b(E)$.

Theorem 4.3. *Assume that for any initial measure $\nu \in M_1(E)$, there is a martingale solution to $(\mathcal{L}, \mathcal{D}, \nu)$. $(\mathcal{L}, \mathcal{D})$ is martingale unique if one of the following conditions is satisfied:*

(i) for some $\lambda_0 > 0$ and for all $\lambda \ge \lambda_0$, $(\lambda - \mathcal{L})(\mathcal{D})$ is separating on E;

(ii) for some $\lambda_0 > 0$ and for all $\lambda \ge \lambda_0$ and $\nu \in M_b^+(E)$,

$$\langle \mu, (\lambda - \mathcal{L})f \rangle = \langle \nu, f \rangle, \ \forall f \in \mathcal{D}$$

has at most one solution $\mu \in M_b^+(E)$;

(iii) $(\mathcal{L}, \mathcal{D})$ is KFM-unique (see Definition 3.3).

The following result should be recognized true by every specialist, but the author has not found it in an exact reference, with some surprise.

Proposition 4.4. *Let $\mathcal{L}$ be a single valued linear operator with domain $\mathbb{D}(\mathcal{L}) = \mathcal{D}$. Assume that $\sigma(\mathcal{D}) = \mathcal{B}$, $\mathcal{D}$ separates the points of E (that is strictly weaker than saying that $\mathcal{D}$ is separating on E) and there is $\{f_n\}_{n \in 0} \subset \mathcal{D}$ such that $\mathcal{D}$ is contained in the closure of $\{F(f_0, \cdots, f_n);\ n \ge 0, F \in C(\mathbb{R}^{n+1})\}$ in $(b\mathcal{B}, \|\cdot\|)$.*

If $(\mathcal{L}, \mathcal{D})$ is martingale unique, then $(\mathcal{L}, \mathcal{D})$ is Markov unique.

5 Uniqueness of semigroup of pur jumps type

In this section we study thoroughly the uniqueness of the following operator:

$$\tilde{\mathcal{L}}^V f(x) = \int_E J(x, dy)(f(y) - f(x)) - V(x)f(x) \qquad (5.1)$$

for all real measurable functions f so that $J|f| < +\infty$, where

(C1) $J(x, dy)$ is a nonnegative kernel on E such that $J(x, \{x\}) = 0$ and $J(x, E) < +\infty$ for all $x \in E$ (it can be interpreted as the jumps rate);

(C2) the potential V is a real measurable function on E such that $V \geq -\kappa$ for some $\kappa \geq 0$.

(C3) There is $(E_n \in \mathcal{B})_{n\in\mathbb{N}} \uparrow E$ such that $J(\cdot, E_n) + |V|1_{E_n}$ is bounded and for any $f \in \mathcal{D}$, there is some n so that $[f \neq 0] \subset E_n$, and

(C4) for each n, $\mathcal{D}_n := \{f \in \mathcal{D};\ [f \neq 0] \subset E_n\}$ is separating on E_n.

Conditions (C3) and (C4) mean that $\tilde{\mathcal{L}}^V$ is observed only "*locally*".

5.1 Probabilistic construction of the "minimal" nonnegative semigroup

We begin with the case where $V = 0$ and we write $\mathcal{L} = \tilde{\mathcal{L}}|_{\mathcal{D}}$.

Consider

1. the Markov kernel $Q(x, dy) := J(x, E)^{-1} J(x, dy)$
2. $\mathbb{Q}_x$: the probability measure on $E^{\mathbb{N}}$ such that its coordinates system $(Y_n)_{n\in\mathbb{N}}$ is a Markov chain with transition Q starting from $x \in E$.
3. the product probability measure $\mathbb{P}_x := \mathbb{Q}_x \otimes \gamma^{\mathbb{N}}$ on $\Omega := E^{\mathbb{N}} \times (\mathbb{R}^+)^{\mathbb{N}}$ where γ is the standard exponential law with parameter 1.

For any $\omega = (Y_n, \gamma_n)_{n\in\mathbb{N}} \in \Omega$, define

$$\begin{aligned} &T_0 = 0,\ T_n(\omega) := \sum_{k=1}^{n} \frac{\gamma_k}{J(Y_{k-1}, E)},\ \forall n \geq 1; \\ &X_t := Y_n,\ \forall t \in [T_n, T_{n+1}); \\ &X_t := \partial,\ \forall t \geq T_\infty := \sup_{n\geq 1} T_n \end{aligned} \tag{5.2}$$

Let $P_t f(x) := \mathbb{E}^x f(X_t) 1_{t<T_\infty}$ whose full generator $\hat{\mathcal{L}}$, by the Lemma below, extends $\mathcal{L}$.

In the case where $V \neq 0$ satisfies (C2), the following Feynman-Kac formula

$$P_t^V f(x) := \mathbb{E}^x 1_{[t<T_\infty]} f(X_t) \exp\left(-\int_0^t V(X_s) ds\right) \tag{5.3}$$

defines a semigroup of nonnegative kernels satisfying $\|P_t^V\| \leq e^{\kappa t} < +\infty$, then belonging to the class $\mathcal{SG}_K$.

Lemma 5.1. *Assume (C1), (C2). Let* $\hat{\mathcal{L}}^V$ *be the full generator of* (P_t^V).

(a) For any $f : E \to \mathbb{R}$ *such that* $J|f| < +\infty$, $M_{t\wedge T_n}^V(f)$ *is a* $\mathbb{P}_x$*-closed martingale for every* $x \in E$ *and* $n \geq 1$, *where*

$$M_t^V(f) := 1_{t<T_\infty} f(X_t) D_t - f(x) - \int_0^t D_s \tilde{\mathcal{L}}^V f(X_s) 1_{s<T_\infty} ds \quad (5.4)$$

and $D_t := \exp\left(-\int_0^t V(X_s)ds\right)$.

(b) Given $f \in b\mathcal{B}$.

(b.i) If $f \in \mathbb{D}(\hat{\mathcal{L}}^V)$, *then* $\hat{\mathcal{L}}^V f(x) = \tilde{\mathcal{L}}^V f(x), \forall x \in E$. *In particular* $\hat{\mathcal{L}}^V$ *is single valued. Moreover* $(\mathcal{L}^V, \mathcal{D}) \subset \hat{\mathcal{L}}^V$ *for* $\mathcal{D}$ *satisfying (C3) and (C4).*

(b.ii) Inversely assume that

$$\mathbb{P}_x\left(\int_0^{T_\infty} (1+V^+)(X_s)ds = +\infty\right) = 1, \ \forall x \in E. \quad (5.5)$$

If $\tilde{\mathcal{L}}^V f \in b\mathcal{B}$, *then* $M_t^V(f)$ *is a* $\mathbb{P}_x$*-martingale for each* $x \in E$, *and* $f \in \mathbb{D}(\hat{\mathcal{L}}^V)$ *and* $\hat{\mathcal{L}}^V f = \tilde{\mathcal{L}}^V f$.

We shall see that the probabilistic condition (5.5) is also necessary for the identification $\hat{\mathcal{L}}^V = \tilde{\mathcal{L}}^V \bigcap (b\mathcal{B} \times b\mathcal{B})$.

Throughout this section, conditions (C1)-(C4) are assumed.

5.2 KF-uniqueness: a Lyapunov function criterion

Besides the criteria in Theorem 3.2 for the KF-uniqueness, we have the following very practical criterion.

Proposition 5.2. *Let $(J, V, \mathcal{D})$ satisfy (C1)-(C4). If there is some measurable function ϕ, strictly positive everywhere on E, such that*

$$\sup_{x\in E}[J\phi(x)] < +\infty, \text{ and } \tilde{\mathcal{L}}^V\phi \leq c\phi \text{ for some } c > 0, \tag{5.6}$$

then $(\mathcal{L}^V, \mathcal{D})$ is KF-unique (see its definition in Theorem 3.2).

One Application:

the KF-uniqueness of the Glauber dynamics associated with continuous gas in finite volume.

5.3 Uniqueness of the Kolmogorov backward equation

Theorem 5.3. *Assume (C1), (C2). Then the following properties are equivalent:*

(i) The abstract Cauchy problem associated with $\tilde{\mathcal{L}}^V$ in $b\mathcal{B}$ has a unique solution;

(ii) $\hat{\mathcal{L}}^V = \tilde{\mathcal{L}}^V \bigcap (b\mathcal{B} \times b\mathcal{B})$;

(iii) for some $\lambda > \kappa$, if $f \in b\mathcal{B}$ satisfies $(\lambda - \tilde{\mathcal{L}}^V)f = 0$, then $f = 0$.

(iv) for some $\lambda > \kappa$, if $f \in b\mathcal{B}^+$ satisfies $(\lambda - \tilde{\mathcal{L}}^V)f = 0$, then $f = 0$.

(v) Condition (5.5) is satisfied, i.e.,

$$\mathbb{P}_x\left(\int_0^{T_\infty}(1+V^+)(X_s)ds = +\infty\right) = 1, \ \forall x \in E.$$

Notice that the equivalence between (iv) and (v) is proved in Hou and Guo [7] (1978), cf. [1], Remarks 3.9. Here we shall give a direct proof.

Corollary 5.4. *In the context of Theorem 5.3, the properties therein are equivalent to any one of*

(vi) (P_t^V) is the unique semigroup of kernels on $b\mathcal{B}$ in the class $\mathcal{SG}_K$ such that its full generator is contained in $\tilde{\mathcal{L}}^V \bigcap (b\mathcal{B} \times b\mathcal{B})$;

(vii) (P_t^V) is the unique semigroup of nonnegative kernels in the class $\mathcal{SG}_K^+(1, \kappa)$, completely regular, such that its full generator is contained in $\tilde{\mathcal{L}}^V \bigcap (b\mathcal{B} \times b\mathcal{B})$.

5.4 KF$^+$-uniqueness

Theorem 5.5. *For the pure jumps operator $(\mathcal{L}^V, \mathcal{D})$ satisfying (C1)-(C4), consider the entrance space*

$$\mathcal{V}^+(\lambda) := \{\mu \in M_b^+(E);\ \langle \mu, (\lambda - \mathcal{L}^V)f\rangle = 0,\ \forall f \in \mathcal{D}\}. \qquad (5.7)$$

the following properties are equivalent:

(i) $(\mathcal{L}, \mathcal{D})$ is KF$^+$-unique (see Def. 3.3);

(ii) $(\mathcal{L}, \mathcal{D})$ is $b\mathcal{B}^+$-unique (see Def. 3.4);

(iii) for some (or equivalently for all) $\lambda > \kappa$, $\mathcal{V}^+(\lambda) = \{0\}$.

5.5 KFM, Markov and martingale uniqueness

Theorem 5.6. *For the pure jumps operator $(\mathcal{L}, \mathcal{D})$ with $V \geq 0$, the following properties are equivalent:*

(a) $(\mathcal{L}^V, \mathcal{D})$ is KFM-unique;

(b) $(\mathcal{L}^V, \mathcal{D})$ is Markov-unique;

(c) one of the following two conditions are satisfied:

(c.i) (P_t^V) is honest, i.e., $P_t^V 1 = 1$ for all $t \geq 0$ (or equivalently $V = 0$ and $\mathbb{P}_x(T_\infty = +\infty) = 1$ for all $x \in E$);

(c.ii) for some (or equivalently for all) $\lambda > 0$, if $\nu \in M_b^+(E)$ verifies

$$\langle \nu, (\lambda - \mathcal{L}^V)f\rangle = 0,\ \forall f \in \mathcal{D},$$

then $\nu = 0$.

(d) $(\mathcal{L}^V, \mathcal{D})$ is martingale-unique;

When $V = 0$, P_t is not honest and (c.ii) holds, it is quite difficult to see why we have the martingale-uniqueness. Following Doob, we can extend our Markov process $(X_t)_{0 \leq t < T_\infty}$ in the following way: on $[T_\infty < +\infty]$ put $X_{T_\infty} = Y$ to be an arbitrary E-valued random variable independent of $(X_t)_{0 \leq t < T_\infty}$ and run the process after time T_∞ as before with Y as initial condition, and so on at and after the second "T_∞", the third...

Why this new honest Markov process *is not* a solution of the martingale problem $(\mathcal{L}, \mathcal{D}, \nu)$? The answer resides at the fact that this new process has a predictable jumps at T_∞, and the jumps of a càglàg solution (X_t) of the martingale problem $(\mathcal{L}, \mathcal{D}, \nu)$ are totally inaccessible (cf. Dellacherie and Meyer).

5.6 Birth-death processes: characterization

Let $E = \mathbb{N}$ and the jumps rate J be given by (for all $i \in \mathbb{N}$)

$J(i, i+1) = b_i > 0$ (birth rate), $J(i, i-1) = a_i > 0$ (death rate), $J(i, j) = 0$, other

where -1 is identified as 0. Consider $E_n = [0, n] \bigcap \mathbb{N}$ and $\mathcal{D} = \{f : \mathbb{N} \to \mathbb{R} / \exists n \in \mathbb{N} : \; f(k) = 0, \forall k > n\}$. Let

$$\mu_0 = 1, \; \mu_k := \frac{b_0 b_1 \cdots b_{k-1}}{a_1 \cdots a_k}, \; \forall k \geq 1. \tag{5.8}$$

Then $\mu(\{k\}) := \mu_k$ is a symmetric measure for $\mathcal{L}$, i.e.,

$$\langle f, \mathcal{L}g \rangle_\mu = \langle \mathcal{L}f, g \rangle_\mu, \; \forall f, g \in \mathcal{D}.$$

And the minimal semigroup (P_t) is symmetric on $L^2(\mathbb{N}, \mu)$ and it is strongly continuous semigroup on $L^p(\mathbb{N}, \mu)$ for all $1 \leq p < +\infty$.

Theorem 5.7. *For the birth-death process above, let*

$$s_0 = \frac{1}{b_0}, \; s_k := \frac{1}{b_k \mu_k} = \frac{1}{b_0} \prod_{j=1}^{k} \frac{a_j}{b_j}, \forall k \geq 1 \; (\textit{ the scale measure}) \tag{5.9}$$

(a) $(\mathcal{L}, \mathcal{D})$ is KF-unique iff $(\mathcal{L}, \mathcal{D})$ is KF$^+$-unique (see Theorem 5.5 for equivalent conditions), and iff

$$\sum_{n=1}^{+\infty} \mu_n \sum_{k=0}^{n-1} s_k = \sum_{k=0}^{\infty} s_k \mu([k+1, +\infty)) = +\infty. \tag{5.10}$$

(This condition means that $+\infty$ is a no entrance boundary, parallel to Feller's classification for one-dimensional diffusion.)

(b) Let $1 < p < +\infty$. $(\mathcal{L}, \mathcal{D})$ is $L^p(\mathbb{N}, \mu)$-unique, i.e., $\mathcal{D}$ is a core for the generator $\mathcal{L}_p$ of (P_t) in $L^p(\mathbb{N}, \mu)$, iff

$$\sum_{n=0}^{+\infty} \mu_n \left(\sum_{k=0}^{n-1} s_k \right)^{p'} = +\infty \tag{5.11}$$

where $p' := p/(p-1)$ is the conjugated number of p.

(c) *The following properties are equivalent.*

(c.i) *The Cauchy problem associated with* $\tilde{\mathcal{L}}$ *is well-posed;*

(c.ii) $(\mathcal{L}, \mathcal{D})$ *is* $L^1(\mathbb{N}, \mu)$*-unique;*

(c.iii) *The semigroup* (P_t) *is honest, i.e.,* $P_t 1 = 1$ *for all* $t \geq 0$.

(c.iv)

$$\sum_{k=0}^{+\infty} s_k \sum_{j=0}^{k} \mu_j = +\infty. \tag{5.12}$$

(d) $(\mathcal{L}, \mathcal{D})$ *is Markov-unique (see Theorem 5.6 for equivalent conditions) iff either (5.12) or (5.10) holds (equivalently,* $(\mathcal{L}, \mathcal{D})$ *is either* $L^1(\mu)$*-unique or KF-unique).*

Though stated differently, the equivalence between (c.iii) and (c.iv) and the so called q-process uniqueness is essentially due to J.K. Zhang (84), cf. Chen [1], Thm. 3.16.

In the case of the presence of a potential V, we have

Theorem 5.8. *Let* $V : \mathbb{N} \to \mathbb{R}$ *be lower bounded,*

$$\tilde{\mathcal{L}}^V f(n) = a_n(f(n-1) - f(n)) + b_n(f(n+1) - f(n)) - V(n)f(n)$$

and $\mathcal{L}^V = \tilde{\mathcal{L}}^V|_{\mathcal{D}}$.

(a) $\mathcal{L}^V$ *is KF-unique iff it is* KF^+*-unique. A sufficient condition for them is*

$$\sum_{n=1}^{+\infty} \mu(n) \left(\sum_{k=0}^{n-1} s_k \sum_{j=0}^{k} \mu_j (1 + V(j)) \right) = +\infty \tag{5.13}$$

(b) *Let* $1 < p < +\infty$. *If*

$$\sum_{n=1}^{+\infty} \mu(n) \left(\sum_{k=0}^{n-1} s_k \sum_{j=0}^{k} \mu_j (1 + V(j)) \right)^{p'} = +\infty \tag{5.14}$$

then $\mathcal{L}^V$ *is* $L^p(\mu)$*-unique (which is equivalent to the essential self-adjointness of* $\mathcal{L}^V$ *in* $L^2(\mu)$ *when* $p = 2$*).*

(c) The following properties are equivalent:

(c.i) The Cauchy problem associated with $\tilde{\mathcal{L}}^V$ *is well-posed;*

(c.ii) $(\mathcal{L}^V, \mathcal{D})$ *is* $L^1(\mathbb{N}, \mu)$*-unique;*

(c.iii) $\mathbb{P}_k\left(\int_0^{T_\infty}(1+V)(X_s)ds = +\infty\right) = 1$ *for all* $k \in \mathbb{N}$*;*

(c.iv)

$$\sum_{k=0}^{+\infty} s_k \sum_{j=0}^{k} \mu_j(1+V(j)) = +\infty. \tag{5.15}$$

Remarks 5.9. It is quite fortunate that the probabilistic condition (c.iii) above (see Theorem 5.6) admits a simple characterization (c.iv). However for $L^p(\mu)$-uniqueness with $1 < p < \infty$, I believe that our sufficient conditions in part (b) of Theorem 5.8 is not necessary, by comparison with the classical Weil's criterion of limit point-limit circle for the essential self-adjointness of the Schrödinger operator $-d^2/dx^2 + V$ on $L^2((0,+\infty), dx)$ (cf. Reed and Simon [?], Theorem X.10 and Theorem X.7). We recall this result for clarifying the situation: letting $V : (0,+\infty) \to \mathbb{R}^+$ be continuous, and bounded near $+\infty$,

1) if $V(x) \geq (3/4)x^{-2}$ near 0, then $d^2/dx^2 - V$ with domain $C_0^\infty(0,+\infty)$ is essentially self-adjoint on $L^2((0,+\infty), dx)$ (or equivalently it is $L^2((0,+\infty), dx)$-unique);

2) if $V(x) \leq (3/4 - \varepsilon)x^{-2}$ near 0 for some $\varepsilon \in (0, 3/4)$, then $d^2/dx^2 - V$ with domain $C_0^\infty(0,+\infty)$ is NOT essentially self-adjoint on $L^2((0,+\infty), dx)$ (or equivalently it is not $L^2((0,+\infty), dx)$-unique).

This criterion is very sensible w.r.t. the constant factor of V, however the sufficient condition in part (b) of Theorem 5.8 is not at all sensible w.r.t. constant factor of V, so it should be non-necessary.

We develop now several corollaries and concrete examples to illustrate the differences between the different notions of uniqueness.

Remarks 5.10. For the birth-death process, we have by Theorem 5.7,

(i) $(\mathcal{L}, \mathcal{D})$ is KF-unique iff $(\mathcal{L}, \mathcal{D})$ is $L^\infty(\mu)$-unique;

(ii) If $\sum_{n\geq 0} \mu_n = +\infty$, then $(\mathcal{L}, \mathcal{D})$ is KF-unique (or $L^\infty(\mu)$-unique), and $L^p(\mu)$-unique for all $1 < p < +\infty$ (by Theorem 5.7(a) and (b)).

(iii) In the case where the symmetric measure μ is finite, i.e., $\sum_{n\geq 0} \mu_n < +\infty$, if $(\mathcal{L}, \mathcal{D})$ is $L^q(\mu)$-unique for some $q \in (1, +\infty]$, then $(\mathcal{L}, \mathcal{D})$ is $L^p(\mu)$-unique for all $1 \leq p < q$ (that follows by Theorem 5.7(b) when $p > 1$ and by what are recalled at the beginning of the proof of Theorem 5.7(b) for $p = 1$).

The finiteness of the symmetric measure μ in the theory of irreducible Markov chains means the positive recurrence, so one might have the impression that (P_t) should be honest in such case. This impression is not correct! Indeed given such μ, we can take $b_k = (k+1)^2/\mu_k$ (and determine $a_k = 1/(s_{k-1}\mu_k) = k^2/\mu_k$ by (5.8)). We have

$$\sum_{k=0}^{+\infty} s_k \sum_{j=0}^{k} \mu_j \leq \mu(\mathbb{N}) \sum_{k=0}^{+\infty} \frac{1}{(k+1)^2} < +\infty$$

and then (P_t) is not honest.

Where is question? The reason is : the finiteness of the symmetric measure μ of (P_t) implies the positive recurrence iff (P_t) is honest!

(iv) By Theorem 5.7(a) and (b), the KF- or KF$^+$-uniqueness of $(\mathcal{L}, \mathcal{D})$ is stronger than its $L^p(\mu)$-uniqueness for every $1 < p < +\infty$ (since (5.10) is stronger than (5.11)).

(v) It is easy to see that $(\mathcal{L}, \mathcal{D})$ is not martingale unique (or equivalently not Markov unique) iff $\sum_{n\geq 0} \mu_n$ and $\sum_{n\geq 0} s_n$ are both finite. Hence if $(\mathcal{L}, \mathcal{D})$ is not martingale unique, it is not $L^p(\mu)$-unique for every $1 \leq p \leq +\infty$.

Corollary 5.11. *For the birth death process, if* $(\mathcal{L}, \mathcal{D})$ *is* $L^p(\mu)$*-unique or martingale unique, where* $1 \le p \le +\infty$ *(*$L^\infty(\mu)$*-uniqueness* $\Leftrightarrow$*KF-uniqueness), then so is* $(\mathcal{L}^V, \mathcal{D})$ *for every* $V : \mathbb{N} \to \mathbb{R}$ *lower bounded.*

This corollary shows an essential difference of the different uniqueness here from the so called q-process uniqueness (cf. [1]): if $V \ge 0$ is not upper bounded, $(\mathcal{L}^V, \mathcal{D})$ is never q-process unique for any choice of $\mathcal{L}$.

Proof. It follows by Theorem 5.7 and Theorem 5.8. □

Corollary 5.12. *(a)* $(\mathcal{L}, \mathcal{D})$ *is KF-unique (or equivalently KF*$^+$*-unique) if*

$$\sum_{n=0}^{+\infty} \frac{1}{a_n} = +\infty,$$

and the inverse is true if $\limsup_{n\to\infty}(b_n/a_{n+1}) < 1$.

(c) Let $p = 1$. $(\mathcal{L}, \mathcal{D})$ *is* $L^1(\mu)$*-unique (or equivalently* (P_t) *is honest) if*

$$\sum_{n=0}^{+\infty} \frac{1}{b_n} = +\infty,$$

and the inverse is true if $\limsup_{n\to\infty}(a_n/b_n) < 1$.

Example 5.13. Let $a_n = a(n+1)^\alpha$ and $b_n = b$ where $a, b > 0$ and $\alpha \in \mathbb{R}$. By Corollary 5.12 (a), $(\mathcal{L}, \mathcal{D})$ is KF-unique iff $\alpha \le 1$. But by Corollary 5.12(b), $(\mathcal{L}, \mathcal{D})$ is always $L^1(\mu)$-unique ($\Leftrightarrow$ the KB-uniqueness of A given in Theorem 5.7(c)). Hence when $\alpha > 1$, we have the KB-uniqueness, but not the KF-uniqueness.

Example 5.14. Let $a_n = a$ and $b_n = b(n+1)^\alpha$ where $a, b > 0$ and $\alpha \in \mathbb{R}$. By Corollary 5.12 (b), $(\mathcal{L}, \mathcal{D})$ is $L^1(\mu)$-unique (equivalent to $P_t 1 = 1$) iff $\alpha \le 1$. But $(\mathcal{L}, \mathcal{D})$ is always KF-unique by Corollary 5.12(a). Hence when $\alpha > 1$, we have the KF-uniqueness, but not the KB-uniqueness.

Example 5.15. Let $a_n = b_n = (n+1)^\alpha$ where $\alpha \in \mathbb{R}$, then $\mu_n = (n+1)^{-\alpha}$ and $s_n = 1$ for all $n \in \mathbb{N}$. $(\mathcal{L}, \mathcal{D})$ is $L^1(\mu)$-unique for all

$\alpha \in \mathbb{R}$ for $\sum_k s_k = +\infty$. For the $L^p(\mu)$-uniqueness, note that

$$\sum_{n=1}^{\infty} \mu_n \left(\sum_{k=0}^{n-1} s_k \right)^{p'} = \sum_{n=1}^{\infty} \frac{n^{p'}}{(n+1)^{\alpha}}$$

is infinite iff $\alpha \leq p' + 1$. By Theorem 5.7, for $p \in [1, \infty]$, $(\mathcal{L}, \mathcal{D})$ is $L^p(\mu)$-unique iff $\alpha \leq p' + 1$ where $p' = p/(p-1)$. In particular when $\alpha = 3$, $(\mathcal{L}, \mathcal{D})$ is $L^p(\mu)$-unique iff $p \in [1, 2]$.

Example 5.16. Let $a_n = n^2(n+1)^2$ and $b_n = (n+1)^4$. Then $\mu_n = (n+1)^{-2}$ and $s_n = (n+1)^{-2}$. As $\sum_{n\geq 0} \mu_n$ and $\sum_{n\geq 0} s_n$ are both finite, $(\mathcal{L}, \mathcal{D})$ is not Markov-unique, not $L^p(\mu)$-unique $(1 \leq p \leq +\infty)$ (then not unique in any sense defined in this paper).

References

[1] M.F. Chen, From Markov chains to ..., World Scientific 1993

[2] E. B. Dynkin, *Markov Processes, Vol. I and II*, Springer-Verlag, Berlin-Göttingen-Heideberg 1965

[3] A. Eberle: Uniqueness and non-uniqueness of singular diffusion operators. Ph.D Dissertation, Bielefeld 1997. LNM

[4] S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence*, John-Wiley & Sons 1986

[5] W. Feller, The parabolic differential equations and the associated semigroups of transformations, *Ann. Math., Vol. 55, No.3, 468-519*(1952)

[6] W. Feller, Semi-groupsof transformations in general weak topologies, *Ann. Math., Vol. 57, 287-308*(1953)

[7] Z.T. Hou and Q.F. Guo, *Time-Homogeneous Markov Processes with countable states space* (in Chinese), Beijing Sci. Press 1978. English Version, Beijing Sci. Press and Springer-Verlag 1988.

[8] L.C.G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingale*, Vol. 1 and 2, Cambridge University Press (2000).

[9] D.W. Stroock and S.R.S. Varadhan, *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin-Heideberg-New York 1979

[10] L. Wu, Uniqueness of Schrödinger operators restricted to a domain, *J. Funct. Anal.* **153**, *No.2, 276-319.*

[11] L. Wu, Uniqueness of Nelson's diffusions, *Probability Theory and Related Fields* **114**, *p549-585 (1999)*

[12] L. Wu and Y. Zhang, Existence and uniqueness of C_0-semigroup in L^∞: a new topological approach, *C.R.A.S. Sér. I. 334, 699-704*(2002).