# Quasi-factorization of $I_\alpha$ and Latała-Oleszkiewicz's inequality for Gibbs random fields

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# outline

- Background.
- Main results.
- Keys of proofs.

# Background

## 1.Latała-Oleszkiewicz's inequality

R.Latała, and K. Oleszkiewicz, Between Sobolev and Poincaré. Lecture Notes in Math. Vol. 1745, 147-168, 2000.

$$\sup_{p\in[1,2)} \frac{\int_{\mathbb{R}} f^2 d\mu - \left(\int_{\mathbb{R}} |f|^p d\mu\right)^{2/p}}{(2-p)^\alpha} \le C \int_{\mathbb{R}} f'^2 d\mu, \quad (1)$$

where $r \in [1,2]$, $\alpha = 2(r-1)/r$, probability measure $\mu(dx) = C_r \exp(-|x|^r)dx$ on $\mathbb{R}$,

# Backgroud

For generality probability space $(\Omega, \mathscr{F}, \mu)$

$$\alpha \in [0,1], I_\alpha(f) := \sup_{p \in [1,2)} \frac{\mu(f^2) - \mu(|f|^p)^{2/p}}{(2-p)^\alpha}$$

$$I_\alpha(f) \leq C D(f,f) \qquad f \in \mathcal{D}(D), \tag{2}$$

where $(D, \mathcal{D}(D))$ the associated Dirichlet form.

# Backgroud

remark1 When $\alpha = 0$, $I_0 = Var(f)$. this inequality coincides with Poincaré inequality; when $\alpha = 1$, $Ent(f^2)/2 \leq I_1 \leq Ent(f^2)$, it is equivalent to the log-Sobolev inequality with different constants. This inequality is more stronger for large $\alpha$.

# Backgroud

Wang, F. Y., A Generalization Poincaré and log-Sobolev Inequalities. Preprint.
F.Barthe, C.Roberto, Sobolev inequalities for probability measures on the real line. Studia mathematica 159(3)(2003)
Wang Feng, PH.D. Thesis
present some criteria of this inequality and characterization of probability measures of this inequality on the line.

# Background

this inequality possesses factorization property , $(\Omega_1, \mathscr{F}_1, \mu_1)(D_1, \mathcal{D}_1(D_1))$, $(\Omega_2, \mathscr{F}_2, \mu_2)(D_2, \mathcal{D}_2(D_2))$

$$D(f,f) := \int_{\Omega_1} D_2(f(x_1),\cdot),f(x_1),\cdot))d\mu_1$$

$$+ \int_{\Omega_2} D_1(f(x_2),\cdot),f(x_2),\cdot))d\mu_2$$

this inequality is remain meaningful and valid in infinite dimensions.

# Backgroud

question1 find conditions under which this inequality is satisfied for given Gibbsian specification ,uniformly in the volume and the boundary condition?

- non interaction, the Gibbs measure is just a product of simple factors.
- interaction, if the interaction is weak, "almost"product.conjecture the conclusion is the same as in the product case.

# Backgroud

## 2.Quasi-factorization of $I_\alpha$

$\mathscr{F}_1, \mathscr{F}_2$ be two sub-$\sigma$-algebras of $\mathscr{F}$

$\mu_i(f) := \mu(f|\mathscr{F}_i), i = 1, 2.$

- $Var_i(f) := \mu_i(f^2) - \mu_i(f)^2$
- $Ent_i(f) := \mu_i(f \log f) - \mu_i(f) \log \mu_i(f) \;\; f \geq 0$
- $I_\alpha^i(f) := \sup_{p\in[1,2)} \frac{\mu(f^2)-\mu(|f|^p)^{2/p}}{(2-p)^\alpha}$

# Backgroud

if $\mathscr{F}_1$ and $\mathscr{F}_2$ independent

- $Var(f) \leq \mu[Var_1(f) + Var_2(f)]$
- $Ent(f) \leq \mu[Ent_1(f) + Ent_2(f)]$
- $I_\alpha(f) \leq \mu[I_\alpha^1(f) + I_\alpha^2(f)]$

remark 2 It is natural to guess that inequalities are stable against appropriate "perturbation" of the hypothesis of independence of the $\sigma$-algebras $\mathscr{F}_1, \mathscr{F}_2$.

# Backgroud

L.Bertini, N.Canerini, F.Cesi, The spectral gap for a Glauber-type dynamics in a continuous gas.Ann IH Poincaré-Probab. stat PR 38.191-108, 2002.

**Proposition 1** Assume that for some $\varepsilon \in [0, \sqrt{2}-1)$, $q \in [1, \infty]$, if

$$\|\mu_1(g) - \mu(g)\|_q \leq \varepsilon \|g\|_q, \quad \forall g \in L^q(\Omega, \mathscr{F}_2, \mu)$$

$$\|\mu_2(g) - \mu(g)\|_q \leq \varepsilon \|g\|_q, \quad \forall g \in L^q(\Omega, \mathscr{F}_1, \mu) \quad (3)$$

then

# Backgroud

$$Var(f) \le (1-2\varepsilon-\varepsilon^2)^{-1}\mu[Var_1(f)+Var_2(f)]. \quad (4)$$

F.Cesi , Quasi-factorization of the entropy and logarithmic Sobolev inequalities for Gibbs random fields. Prob. Theo. Rel. Fie. 120. 569-584, 2001.

Proposition 2 There exist $m < \infty$, $\theta : [0,1) \mapsto \mathbb{R}_+$, $\limsup_{\varepsilon\to 0}(\theta(\varepsilon)/\varepsilon) \le m$, if some $\varepsilon \in [0,1)$,

$$\|\mu_1(g) - \mu(g)\|_\infty \le \varepsilon\|g\|_1 \quad \forall g \in L^1(\Omega, \mathscr{F}_2, \mu) \quad (5)$$

# Backgroud

then

$$Ent(f^2) \le \mu[Ent_1(f^2) + Ent_2(f^2)] + \theta(\varepsilon)Var(f),$$

$$Ent(f^2) \le \mu[Ent_1(f^2) + Ent_2(f^2)] + \theta(\varepsilon)Ent(f^2). \tag{6}$$

question 2 under what is "weak dependence" condition, $I_\alpha$ has analogous results?

# main results

**Proposition 3** For some $\varepsilon \in [0, 1/16]$, if

$$\|\mu_2(g) - \mu(g)\|_\infty \leq \varepsilon \|g\|_1 \quad \forall g \in L^1(\Omega, \mathscr{F}_1, \mu) \tag{7}$$

then

$$I_\alpha(f) \leq \mu[I_\alpha^1(f) + I_\alpha^2(f)] + 16\varepsilon Var(f). \tag{8}$$

# main results

As an application under some condition , Gibbs specification with translation invariant and finite range summable interaction has uniform Latała-Oleszkiewicz's inequalities.

Gibbs measures. Single spin space $(S, \mathscr{E}, \nu)$

$\mathbb{Z}^d$ configuration space$(\Omega, \mathscr{F}) := (S^{\mathbb{Z}^d}, \mathscr{E}^{\mathbb{Z}^d})$, finite configuration space

$\forall \Lambda \in \mathbb{F} \subset \mathbb{Z}^d (\Omega_\Lambda, \mathscr{F}_\Lambda) = (S^\Lambda, \mathscr{E}^\Lambda)$ consider a translation invariant ,summable interaction $J$ of finite range $r$

$$(H_1)\ ,\ J_{\Lambda+x} \circ \vartheta_x = J_\Lambda,\ \vartheta_x(\sigma)(y) = \sigma(y-x), x, y \in \mathbb{Z}^d.$$

# Main results

$(H_2)$ when $d(\Lambda,\Lambda) := \sup_{x,y\in\Lambda} d(x,y) > r$, $J_\Lambda = 0$,

$(H_3)$ $\|J\| := \sum_{\Lambda\in\mathbb{F},\Lambda\ni 0} \|J_\Lambda\|_u < \infty$,.

The Hamitonian

$H_\Lambda : \Omega \ni \sigma \longrightarrow \sum_{\Lambda\in\mathbb{F}:A\cap\Lambda\neq\emptyset} J_A(\sigma) \in \mathbb{R}$.

$\forall\Lambda \in \mathbb{F}, \quad \tau \in \Omega$, $(\Omega_\Lambda, \mathscr{F}_\Lambda)$, $H^\tau_\Lambda(\sigma) := H_\Lambda(\sigma_\Lambda\tau_{\Lambda^c})$,

$$\mu^\tau_\Lambda(d\sigma) := (Z^\tau_\Lambda)^{-1} \exp\left[-H^\tau_\Lambda(\sigma)\right]\nu^\Lambda(d\sigma_\Lambda) \qquad (9)$$

# Main results

"generalized" Dirichlet form $\mathcal{E}_\Lambda^\tau$,
we need are the following properties of $\mathcal{E}$:

$(E_1)$ exists a set $\mathscr{A}$ of measurable functions which is domain for all $\mathscr{A} \subset \{\mathcal{E}_\Lambda^\tau : \Lambda \in \mathbb{F}, \tau \in \Omega\}$
$\mathcal{E}_\Lambda^\tau : f \in \mathscr{A} \longmapsto \mathbb{R}^+$.

$(E_2)$ $\forall V \subset \Lambda$, $\tau \in \Omega$, $\forall f \in \mathscr{A}$, the function $\mathcal{E}_V(f) \in L^1(\mu_\Lambda^\tau)$.

# Main results

"generalized" Dirichlet form $\mathcal{E}^{\tau}_{\Lambda}$,
we need are the following properties of $\mathcal{E}$:

$(E_1)$ exists a set $\mathscr{A}$ of measurable functions which is domain for all $\mathscr{A} \subset \{\mathcal{E}^{\tau}_{\Lambda} : \Lambda \in \mathbb{F}, \tau \in \Omega\}$ $\mathcal{E}^{\tau}_{\Lambda} : f \in \mathscr{A} \longmapsto \mathbb{R}^{+}$.

$(E_2)$ $\forall V \subset \Lambda$, $\tau \in \Omega$, $\forall f \in \mathscr{A}$, the function $\mathcal{E}_V(f) \in L^1(\mu^{\tau}_{\Lambda})$.

# Main results

$(E_3)$ if $\Lambda = V_1 \cup V_2$, then
$\mu_\Lambda^\tau[\mathcal{E}_{V_1}(f) + \mathcal{E}_{V_2}(f)] = \mathcal{E}_\Lambda^\tau(f) + \mu_\Lambda^\tau(\mathcal{E}_{V_1 \cap V_2}(f))$.

for all $\Lambda \in \mathbb{F}$, define Latała-Oleszkiewicz constant $\beta_{\alpha,\Lambda} \in [0, \infty]$ as the infimum of all positive real numbers $c$ such that

$$I_{\Lambda,\alpha}^\tau(f) \leq c\mathcal{E}_\Lambda^\tau(f)$$

$(E_4)$ The quantity $\beta_{\alpha,\Lambda}$ is finite for all $\Lambda \in \mathbb{F}$.

# Main results

hypothesis on J is the following:
Assumtion(CA)(Complete analyticity)：There exist $m, C > 0$ such that for all $V \in \mathbb{F}$, $\forall x \in \partial_r^+ V$, $\triangle \subset V$,for all $\sigma, \omega \in \Omega$ with $\sigma(x) = \omega(y)$,if $x \neq y$ then

$$\|\frac{\rho_{V,\triangle}^{\omega}}{\rho_{V,\triangle}^{\sigma}} - 1\|_u \leq Ce^{-md(\triangle,y)}$$

# main results

where, $\mu_\Lambda^\tau$ as the restriction of $\mu_{\Lambda,\triangle}^\tau$ on $\Omega_\triangle$.$V \subset \Lambda$,

$$\rho_{\Lambda,V}^\tau(\sigma) := (Z_\Lambda^\tau) \int_{\Omega_{\Lambda\setminus V}} \exp\left[-H_\Lambda^\tau(\eta_{\Lambda\setminus V}\sigma_V)\right]\nu^{\Lambda\setminus V}(d\eta)$$

theorem 1 let $J$ be a translation invariant ,summable interaction of finite range $r$ such that assumptions(CA) holds,, and let $\{\mathcal{E}_\Lambda^\tau : \Lambda \in \mathbb{F}, \tau \in \Omega\}$satisfy condition$(E_1), \cdots, (E_4)$ then

$$\sup_{\Lambda \in \mathbb{F}} \beta_{\alpha,\Lambda} < \infty$$

# Keys of proofs

lemma 1 For all $f \geq 0,\ f \in L^2(\mu)$, then

$$Var_p(f) \leq \mu[Var_p^1(f) - Var_p^2(f)]$$

$$+2\left[\mu(\mu_1\mu_2(f^p)^{2/p-1}f^p) - \mu(f^p)^{2/p}\right]/p.$$

lemma 2 For $p \in [1,2)$, $f \geq 0$, $f \in L^2(\mu)$, if satisfy$(7)$, then

$$\mu(\mu_1\mu_2(f^p)^{2/p-1}f^p) - \mu(f^p)^{2/p} \leq 8\varepsilon(2-p)/pVar(f).$$

# Keys of proofs

proof of the proposition 3, lemma 1 and lemma 2.

proof of the theorem 1： proposition 3 and iterative procedure to Latała-Oleszkiewicz constant $\beta_{\alpha,\Lambda}$.

# THE END—-THANK YOU!