Strong Ergodicity: Some New Results

Yong-Hua MAO

Beijing Normal University

maoyh@bnu.edu.cn

August 10–14, 2004, RCS, BNU

Content

- Definition
- Coupling methods
- Convergence rate: diffusion process
- Convergence rate: Q process
- Essential spectrum

Definition

$P_t(x,\cdot)$: Markov kernel with stationary distribution π
P_t: Markov semigroup with generator L

- Definition:

$$\sup_x ||P_t(x,\cdot) - \pi||_{\mathrm{Var}} = ||P_t - \pi||_{\infty\to\infty} \to 0$$

By semigroup property, define the exponential convergence rate

$$\alpha = \sup\left\{\epsilon > 0 : \exists C < \infty \ s.t. \ ||P_t - \pi||_{\infty\to\infty} \le Ce^{-\epsilon t}\right\}$$

that is

$$\alpha = -\lim_{t\to\infty} \frac{1}{t} \log ||P_t - \pi||_{\infty\to\infty},$$

Recall: qualitative

- Meyn & Tweedie et al: petite set C, $\sup_{x \in E} \mathbb{E}^x \tau_C < \infty$. and the famous drift condition

Recall: qualitative

- Meyn & Tweedie et al: petite set C, $\sup_{x\in E} \mathbb{E}^x \tau_C < \infty$. and the famous drift condition
- H-J, Zhang; An-Yue, Chen; Xiang Lin; Zhenting Hou: strong ergodicity $\Rightarrow$ non FRR semigroup; they are equivalent for monotonic process; birth-death process, extended branching process

Recall: qualitative

- Meyn & Tweedie et al: petite set C, $\sup_{x\in E}\mathbb{E}^x\tau_C < \infty$. and the famous drift condition
- H-J, Zhang; An-Yue, Chen; Xiang Lin; Zhenting Hou: strong ergodicity $\Rightarrow$ non FRR semigroup; they are equivalent for monotonic process; birth-death process, extended branching process
- Zhang,Y-H: single birth process (or upward skip free process)

Recall: convergence rate

- Doeblin(1930-40): Discrete time Markov chain

$$\exists n, P^n(i, A) \geq \epsilon\nu(A) \Rightarrow r_{exp} \leq (1-\epsilon)^{1/n}.$$

Where

$$r_{exp} = \inf\{r \leq 1 : ||P^n - \pi||_{\infty\to\infty} \leq Cr^n\}$$

Recall: convergence rate

- Doeblin(1930-40): Discrete time Markov chain

$$\exists n, P^n(i,A) \geq \epsilon\nu(A) \Rightarrow r_{exp} \leq (1-\epsilon)^{1/n}.$$

  Where

$$r_{exp} = \inf\{r \leq 1 : ||P^n - \pi||_{\infty\to\infty} \leq Cr^n\}$$

- Dobrushin coefficient(1960):

$$\delta(P) = \frac{1}{2}\sup_{ij}\sum_k |p_{ik} - p_{jk}| \Rightarrow r_{exp} \leq \delta(P)$$

Recall: convergence rate

- Doeblin(1930-40): Discrete time Markov chain

$$\exists n, P^n(i, A) \geq \epsilon\nu(A) \Rightarrow r_{exp} \leq (1-\epsilon)^{1/n}.$$

Where

$$r_{exp} = \inf\{r \leq 1 : ||P^n - \pi||_{\infty\to\infty} \leq Cr^n\}$$

- Dobrushin coefficient(1960):

$$\delta(P) = \frac{1}{2}\sup_{ij}\sum_k |p_{ik} - p_{jk}| \Rightarrow r_{exp} \leq \delta(P)$$

- Griffeath(1975): coupling method, basic coupling

Recall: convergence rate

- Doeblin(1930-40): Discrete time Markov chain

$$\exists n, P^n(i, A) \geq \epsilon\nu(A) \Rightarrow r_{exp} \leq (1-\epsilon)^{1/n}.$$

  Where

$$r_{exp} = \inf\{r \leq 1 : ||P^n - \pi||_{\infty\to\infty} \leq Cr^n\}$$

- Dobrushin coefficient(1960):

$$\delta(P) = \frac{1}{2}\sup_{ij}\sum_k |p_{ik} - p_{jk}| \Rightarrow r_{exp} \leq \delta(P)$$

- Griffeath(1975): coupling method, basic coupling
- Aldous; Fill; Diaconis; Saloff-Coste: Markov chains

Recall: convergence rate

- Saloff-Coste: diffusion on compact manifold

Recall: convergence rate

- Saloff-Coste: diffusion on compact manifold
- Hino; Wang, F-Y: Convergence of semigroup in L^1 (by symmetry)

$$||P_t - \pi||_{\infty \to \infty} = ||P_t - \pi||_{1 \to 1}$$

Recall: convergence rate

- Saloff-Coste: diffusion on compact manifold
- Hino; Wang, F-Y: Convergence of semigroup in L^1 (by symmetry)

$$||P_t - \pi||_{\infty \to \infty} = ||P_t - \pi||_{1 \to 1}$$

- Wu, L-M: a comprehensive study in various spaces, discrete time

Recall: convergence rate

- Saloff-Coste: diffusion on compact manifold
- Hino; Wang, F-Y: Convergence of semigroup in L^1 (by symmetry)

$$||P_t - \pi||_{\infty \to \infty} = ||P_t - \pi||_{1 \to 1}$$

- Wu, L-M: a comprehensive study in various spaces, discrete time
- Mao: diffusion and Q process

Coupling

- $\mathbb{X}_t = (X_t^1, X_t^2)$ coupling Markov process. Coupling time: $T = \inf\{t \geq 0 : X_t^1 = X_t^2\}$

Coupling

- $\mathbb{X}_t = (X_t^1, X_t^2)$ coupling Markov process. Coupling time: $T = \inf\{t \geq 0 : X_t^1 = X_t^2\}$
- Basic coupling inequality:

$$||P(t, x_1, \cdot) - \pi||_{\mathrm{Var}} \leq 2 \int_E \pi(dx_2) \mathbb{P}^{x_1, x_2}[T > t]$$

Coupling

- $\mathbb{X}_t = (X_t^1, X_t^2)$ coupling Markov process. Coupling time: $T = \inf\{t \geq 0 : X_t^1 = X_t^2\}$
- Basic coupling inequality:

$$||P(t, x_1, \cdot) - \pi||_{\mathrm{Var}} \leq 2 \int_E \pi(dx_2) \mathbb{P}^{x_1, x_2}[T > t]$$

- Saloff-Coste: variance threshold time

$$\eta = \inf\left\{t \geq 0 : ||P_t - \pi||_{\infty \to \infty} \leq e^{-1}\right\}$$

Coupling

- $\mathbb{X}_t = (X_t^1, X_t^2)$ coupling Markov process. Coupling time: $T = \inf\{t \geq 0 : X_t^1 = X_t^2\}$
- Basic coupling inequality:

$$||P(t, x_1, \cdot) - \pi||_{\mathrm{Var}} \leq 2 \int_E \pi(dx_2)\mathbb{P}^{x_1,x_2}[T > t]$$

- Saloff-Coste: variance threshold time

$$\eta = \inf\left\{t \geq 0 : ||P_t - \pi||_{\infty\to\infty} \leq e^{-1}\right\}$$

- If

$$S := \sup_{x_1,x_2} \mathbb{E}^{x_1,x_2} T < \infty$$

then $\alpha \geq (eS)^{-1}$.

Coupling

The above estimate can be improved as:

- $\exists \lambda > 0$

$$\sup_{x_1, x_2} \mathbb{E}^{x_1, x_2} e^{\lambda T} < \infty,$$

then $\alpha \geq \lambda$.

Coupling

The above estimate can be improved as:

- $\exists \lambda > 0$

$$\sup_{x_1,x_2} \mathbb{E}^{x_1,x_2} e^{\lambda T} < \infty,$$

  then $\alpha \geq \lambda$.

- Usually,

$$\sup_{x_1,x_2} \mathbb{E}^{x_1,x_2} T^n \leq n! S^n, \quad n \geq 1$$

  so that

$$\alpha \geq S^{-1}$$

Upper bound for α

- Right-continuous is assumed τ_A: hitting time of A.

Upper bound for α

- Right-continuous is assumed τ_A: hitting time of A.
- For closed A with $\pi(A) > 0$

$$\sup_{x\geq 0} \mathbb{E}^x \tau_A \leq \inf\left\{\frac{R}{\pi(A) - Ce^{-\alpha R}} : R > \alpha^{-1}\log\frac{C}{\pi(A)}\right\}.$$

In particular,

$$\alpha \leq \frac{2}{\pi(A)}\log\frac{2C}{\pi(A)}\left(\sup_{x\geq 0}\mathbb{E}^x\tau_A\right)^{-1}$$

Convergence rate: diffusion

- (M, g) complete Riemannian (∂M empty or convex)
  $L = \Delta + \nabla V$, $V \in C^2(M)$, $X_t : L$-diffusion,
  $\pi(dx) = e^V dx/Z$
  $\mathrm{Ric}_M \geq -Kg$, $K(V) = \inf\{r : \mathrm{Hess}_V - \mathrm{Ric}_M \leq r\}$.

Convergence rate: diffusion

- (M, g) complete Riemannian (∂M empty or convex)
  $L = \Delta + \nabla V$, $V \in C^2(M)$, $\quad X_t : L$-diffusion,
  $\pi(dx) = e^V dx/Z$
  $\mathrm{Ric}_M \geq -Kg$, $K(V) = \inf\{r : \mathrm{Hess}_V - \mathrm{Ric}_M \leq r\}$.
- For suitable $\gamma \in C[0, D]$ (for example, $\gamma(r) = K(V)r$)

$$C(r) = \exp\left[\int_0^r \gamma(s)ds/4\right], \quad r \in [0, d)$$

$$\delta(M) = \frac{1}{4}\int_0^D C(s)^{-1}ds \int_s^D C(u)du.$$

Convergence rate: diffusion

- (M, g) complete Riemannian (∂M empty or convex)
  $L = \Delta + \nabla V$, $V \in C^2(M)$, $\quad X_t : L$-diffusion,
  $\pi(dx) = e^V dx/Z$
  $\mathrm{Ric}_M \geq -Kg$, $K(V) = \inf\{r : \mathrm{Hess}_V - \mathrm{Ric}_M \leq r\}$.
- For suitable $\gamma \in C[0, D]$ (for example, $\gamma(r) = K(V)r$)

$$C(r) = \exp\left[\int_0^r \gamma(s)ds/4\right], \quad r \in [0, d)$$

$$\delta(M) = \frac{1}{4}\int_0^D C(s)^{-1}ds \int_s^D C(u)du.$$

- $\alpha \geq \delta(M)^{-1}$.

Convergence rate: Q process

- Coupling $\tilde{Q} = (\tilde{q}_{ij})$ for $Q = (q_{ij})$:

$$\tilde{Q}f(x_1, x_2) = Qf(x_1), \quad f(x_1, x_2) = f(x_1)$$
$$\tilde{Q}f(x_1, x_2) = Qf(x_2), \quad f(x_1, x_2) = f(x_2)$$

Convergence rate: Q process

- Coupling $\tilde{Q} = (\tilde{q}_{ij})$ for $Q = (q_{ij})$:

$$\tilde{Q}f(x_1, x_2) = Qf(x_1), \quad f(x_1, x_2) = f(x_1)$$
$$\tilde{Q}f(x_1, x_2) = Qf(x_2), \quad f(x_1, x_2) = f(x_2)$$

- Independent coupling:

$$\tilde{Q}_0 f(x_1, x_2) = [Qf(\cdot, x_2)](x_1) + [Qf(x_1, \cdot)](x_2)$$

Convergence rate: Q process

- Classical coupling

$$\tilde{Q}_c f(i_1, i_2) = \begin{cases} [Qf(\cdot, i_2)](i_1) + [Qf(i_1, \cdot)](i_2), & \text{if } i_1 \neq i_2, \\ Qg(i_1), & \text{if } i_1 = i_2, \end{cases}$$

where $g(i) = f(i, i)$.

Convergence rate: Q process

- Classical coupling

$$\tilde{Q}_c f(i_1, i_2) = \begin{cases} [Qf(\cdot, i_2)](i_1) + [Qf(i_1, \cdot)](i_2), & \text{if } i_1 \neq i_2, \\ Qg(i_1), & \text{if } i_1 = i_2, \end{cases}$$

  where $g(i) = f(i, i)$.

- For probability measures $\mu_i, \nu_i (i = 1, 2)$,

$$||\mu_1 \times \mu_2 - \nu_1 \times \nu_2||_{\text{Var}} \leq ||\mu_1 - \nu_1||_{\text{Var}} + ||\mu_2 - \nu_2||_{\text{Var}}.$$

Convergence rate: Q process

- Classical coupling

$$\tilde{Q}_c f(i_1, i_2) = \begin{cases} [Qf(\cdot, i_2)](i_1) + [Qf(i_1, \cdot)](i_2), & \text{if } i_1 \neq i_2, \\ Qg(i_1), & \text{if } i_1 = i_2, \end{cases}$$

where $g(i) = f(i, i)$.

- For probability measures $\mu_i, \nu_i (i = 1, 2)$,

$$||\mu_1 \times \mu_2 - \nu_1 \times \nu_2||_{\text{Var}} \leq ||\mu_1 - \nu_1||_{\text{Var}} + ||\mu_2 - \nu_2||_{\text{Var}}.$$

- For classical coupling (or independent coupling) with coupling time T, strong ergodicity $\Leftrightarrow \sup_{i_1 i_2} \mathbb{E}^{i_1 i_2} T < \infty$.

Convergence rate: Q process

A sufficient condition:

- Let D be an absorbing set for $Q = (q_{ij})$ process (i.e. $i \in D, q_i = 0$), and $i \in D^c, \sum_{j \in D} q_{ij} \geq \beta$, then $\sup_{i \in D^c} \mathbb{E}^i \tau_D \leq 1/\beta$.

Convergence rate: Q process

A sufficient condition:

- Let D be an absorbing set for $Q = (q_{ij})$ process (i.e. $i \in D, q_i = 0$), and $i \in D^c, \sum_{j \in D} q_{ij} \geq \beta$, then $\sup_{i \in D^c} \mathbb{E}^i \tau_D \leq 1/\beta$.
- For any coupling $\tilde{Q}$, if $\beta_{ij} := \sum_k \tilde{q}_{(ij)(kk)} \geq \beta$, then $\alpha \geq \beta$.

Convergence rate: Q process

- Basic coupling:

$$\tilde{Q}_b f(i_1,i_2) = \sum_j (q^1_{i_1 j} - q^2_{i_2 j})(f(j,i_2) - f(i_1,i_2))$$
$$+ \sum_j (q^2_{i_2 j} - q^1_{i_1 j})(f(i_1,j) - f(i_1,i_2))$$
$$+ \sum_j (q^1_{i_1 j} \wedge q^2_{i_2 j})(f(j,j) - f(i_1,i_2))$$

Convergence rate: Q process

- Basic coupling:

$$\tilde{Q}_b f(i_1, i_2) = \sum_j (q^1_{i_1 j} - q^2_{i_2 j})(f(j, i_2) - f(i_1, i_2))$$
$$+ \sum_j (q^2_{i_2 j} - q^1_{i_1 j})(f(i_1, j) - f(i_1, i_2))$$
$$+ \sum_j (q^1_{i_1 j} \wedge q^2_{i_2 j})(f(j, j) - f(i_1, i_2))$$

- If $i \neq j$

$$\beta_{ij} = q_{ij} + q_{ji} + \sum_{k \neq i,j} q_{ik} \wedge q_{jk} \geq \beta,$$

then $\alpha \geq \beta$.

Convergence rate: Q process

- Example:
  Let $q_{ij} = \pi_j, i \neq j; \pi_j - 1, j = i$, then $\alpha = 1$. Furthermore $\sigma(L|L^\infty) = \sigma(L|L^2) = \{0, 1\}$. Indeed, $\beta_{ij} = \pi_j + \pi_i + \sum_{k \neq i,j} \beta_k = 1$.

Convergence rate: Q process

- Example:
  Let $q_{ij} = \pi_j, i \neq j; \pi_j - 1, j = i$, then $\alpha = 1$. Furthermore $\sigma(L|L^\infty) = \sigma(L|L^2) = \{0, 1\}$. Indeed, $\beta_{ij} = \pi_j + \pi_i + \sum_{k\neq i,j} \beta_k = 1$.

- Example
  Let $q_{0i} = b_i, q_{i0} = q_i, i \neq 0, \sum_i b_i < \infty$, the strong ergodicity $\Leftrightarrow \inf_i q_i > 0$, and $\alpha \geq \inf_i q_i$. Indeed,

$$\beta_{ij} = \begin{cases} q_i + b_i, & j = 0, i > 0 \\ q_j + b_j, & i = 0, j > 0 \\ q_i \wedge q_j, & i > 0, j > 0 \end{cases}$$

  $\alpha \leq \text{gap} \leq \inf_i q_i$, thus $\alpha = \text{gap} = \inf_i q_i$.

Convergence rate: Q process

- Monotonicity and Order-preserving coupling
  $(E, \le)$: partial order, $\mathscr{M}$: class of monotone functions
  Monotonicity: $P_t f(x) \le P_t f(y), \quad x \le y, f \in \mathscr{M}$

Convergence rate: Q process

- Monotonicity and Order-preserving coupling
  $(E, \leq)$: partial order, $\mathscr{M}$: class of monotone functions
  Monotonicity: $P_t f(x) \leq P_t f(y), \quad x \leq y, f \in \mathscr{M}$
- (Zhang,Y-H) Assume $E = \{0, 1, 2, \cdots\}$ with the usual order. For monotone process, there exists coupling $\mathbb{X}_t = (X^1, X^2)$ s.t.

$$\mathbb{P}^{i_1 i_2}[X_t^1 \leq X_t^2] = 1, \quad t \geq 0, i_1 \leq i_2.$$

Convergence rate: Q process

- Monotonicity and Order-preserving coupling
  $(E, \leq)$: partial order, $\mathscr{M}$: class of monotone functions
  Monotonicity: $P_t f(x) \leq P_t f(y), \quad x \leq y, f \in \mathscr{M}$
- (Zhang,Y-H) Assume $E = \{0, 1, 2, \cdots\}$ with the usual order. For monotone process, there exists coupling $\mathbb{X}_t = (X^1, X^2)$ s.t.

$$\mathbb{P}^{i_1 i_2}[X_t^1 \leq X_t^2] = 1, \quad t \geq 0, i_1 \leq i_2.$$

- For a monotone process on $\{0, 1, 2, \cdots\}$, we have

$$\alpha \geq 1 / \sup_i \mathbb{E}^i \tau_0$$

Convergence rate: Q process

Extended branching process with Q matrix as:

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0; \\ -q_0, & j = i = 0; \\ r_i p_0, & j = i-1,\ i \geq 1; \\ r_i p_{k+1}, & j = i+k,\ i,k \geq 1; \\ -r_i(1-p_1), & j = i \geq 1; \\ 0, & \text{else}, \quad i,j \in \mathbf{Z}_+. \end{cases}$$

Assume $r_i = i^\theta, q_{0i} = p_i$
let $M = \sum_k k p_k, P(s) = \sum_{k=0}^{\infty} p_k s^k - s$, then

$$\mathbb{E}^i \tau_0 = \frac{1}{\Gamma(\theta)} \int_0^1 \frac{(1-s^i)|\log s|^{\theta-1} ds}{P(s)}.$$

Convergence rate: Q process

- If $M < 1, \theta > 1$, then

$$\alpha \geq \frac{(1-M)(\theta-1)\Gamma(\theta)}{2(\theta-1)\Gamma(\theta)+1}.$$

Convergence rate: Q process

- If $M < 1, \theta > 1$, then

$$\alpha \geq \frac{(1-M)(\theta-1)\Gamma(\theta)}{2(\theta-1)\Gamma(\theta)+1}.$$

- If $M = 1, \theta > 2$, then

$$\alpha \geq \frac{ab(\theta-2)\Gamma(\theta)}{b(\theta-2)\Gamma(\theta)+4a}.$$

where $a = P(1/2), b = P''(1/2)$.

Discrete spectrum

- In a Banach space $\mathbb{B}$, discrete spectrum $\sigma_{\text{disc}}(L|\mathbb{B})$: eigenvalues with finite multiplicity
  $\sigma_{\text{ess}}(L|\mathbb{B}) = \sigma(L|\mathbb{B}) \setminus \sigma_{\text{disc}}(L|\mathbb{B})$

Discrete spectrum

- In a Banach space $\mathbb{B}$, discrete spectrum $\sigma_{\text{disc}}(L|\mathbb{B})$: eigenvalues with finite multiplicity
  $\sigma_{\text{ess}}(L|\mathbb{B}) = \sigma(L|\mathbb{B}) \setminus \sigma_{\text{disc}}(L|\mathbb{B})$
- (Wu,L-M) For diffusion and Q process, $\sigma_{\text{ess}}(L|L^\infty) = \emptyset$ iff

$$\forall \epsilon > 0, \exists K \subset\subset E, \quad s.t. \sup_x \mathbb{E}^x \tau_K \leq \epsilon$$

  where τ_K is the (first) hitting time of K

Ultracontractivity

Assume the process is reversible

- (weak) Ultracontractivity: (local)

$$\exists t_0 > 0, \quad s.t. \quad ||P_t||_{1\to\infty} < \infty$$

then $\alpha = \mathrm{gap}(L)$.

Ultracontractivity

Assume the process is reversible

- (weak) Ultracontractivity: (local)

$$\exists t_0 > 0, \quad s.t. \quad ||P_t||_{1\to\infty} < \infty$$

  then $\alpha = \mathrm{gap}(L)$.

- (strong) Ultracontractivity: (globe)

$$\forall t > 0, \quad ||P_t||_{1\to\infty} < \infty$$

  then

$$\sigma_{\mathrm{ess}}(L|L^\infty) = \sigma_{\mathrm{ess}}(L|L^2) = \emptyset \quad \sigma_{\mathrm{disc}}(L|L^\infty) = \sigma_{\mathrm{disc}}(L|L^2)$$

Ultracontractivity

- (strong) Ultracontractivity implied by Nash inequality

Ultracontractivity

- (strong) Ultracontractivity implied by Nash inequality
- Example: On $\mathbb{R}^d$, $L = \Delta + \nabla V \cdot \nabla$ with $V(x) \approx -|x|^\gamma (|x| \to \infty), \gamma > 2$.

Ultracontractivity

- (strong) Ultracontractivity implied by Nash inequality
- Example: On $\mathbb{R}^d$, $L = \Delta + \nabla V \cdot \nabla$ with $V(x) \approx -|x|^\gamma(|x| \to \infty), \gamma > 2$.
- Example: Birth-death process: $a_i = b_i = i^\gamma, \gamma > 2, b_0 = 1$

Discrete spectrum: diffusion

- L-Diffusion on Riemannian manifold
  ρ: distance function, D: the diameter

$$h(\gamma, r) = \exp\left(\int_0^r \gamma(u)du\right)$$

$$F(\gamma, D) = \int_0^D h(\gamma, s)^{-1}ds \int_s^D h(\gamma, u)du$$

Discrete spectrum: diffusion

- L-Diffusion on Riemannian manifold
  ρ: distance function, D: the diameter

$$h(\gamma, r) = \exp\left(\int_0^r \gamma(u)du\right)$$

$$F(\gamma, D) = \int_0^D h(\gamma, s)^{-1} ds \int_s^D h(\gamma, u) du$$

- Suppose that $\gamma_1, \gamma_2 \in C[0, \infty)$ such that
  $\gamma_2(\rho)\rho \le L\rho \le \gamma_1(\rho)\rho$, then
  (1) $F(\gamma_1, D) < \infty$ implies $\sigma_{\text{ess}}(L|L^\infty) = \emptyset$
  (2) strong ergodicity implies $F(\gamma_2, D) < \infty$.

Single death process

Single death process (downward skip free process):
$q_{ij} = 0$ for $i - j \geq 2$

- Regularity is assumed

Single death process

Single death process (downward skip free process):
$q_{ij} = 0$ for $i - j \geq 2$

- Regularity is assumed
- For single death process, strong ergodicity is equivalent to $\sigma_{\text{ess}}(L|L^\infty) = \emptyset$

Single death process

Single death process (downward skip free process):
$q_{ij} = 0$ for $i - j \geq 2$

- Regularity is assumed
- For single death process, strong ergodicity is equivalent to $\sigma_{\text{ess}}(L|L^{\infty}) = \emptyset$
- Application to extended branching process to get the explicit criteria.

Random walk on trees

- time–continuous random walk on T

Random walk on trees

- time–continuous random walk on T
- Tree $T = (o, V, E)$: root o, vertices V, edges E.

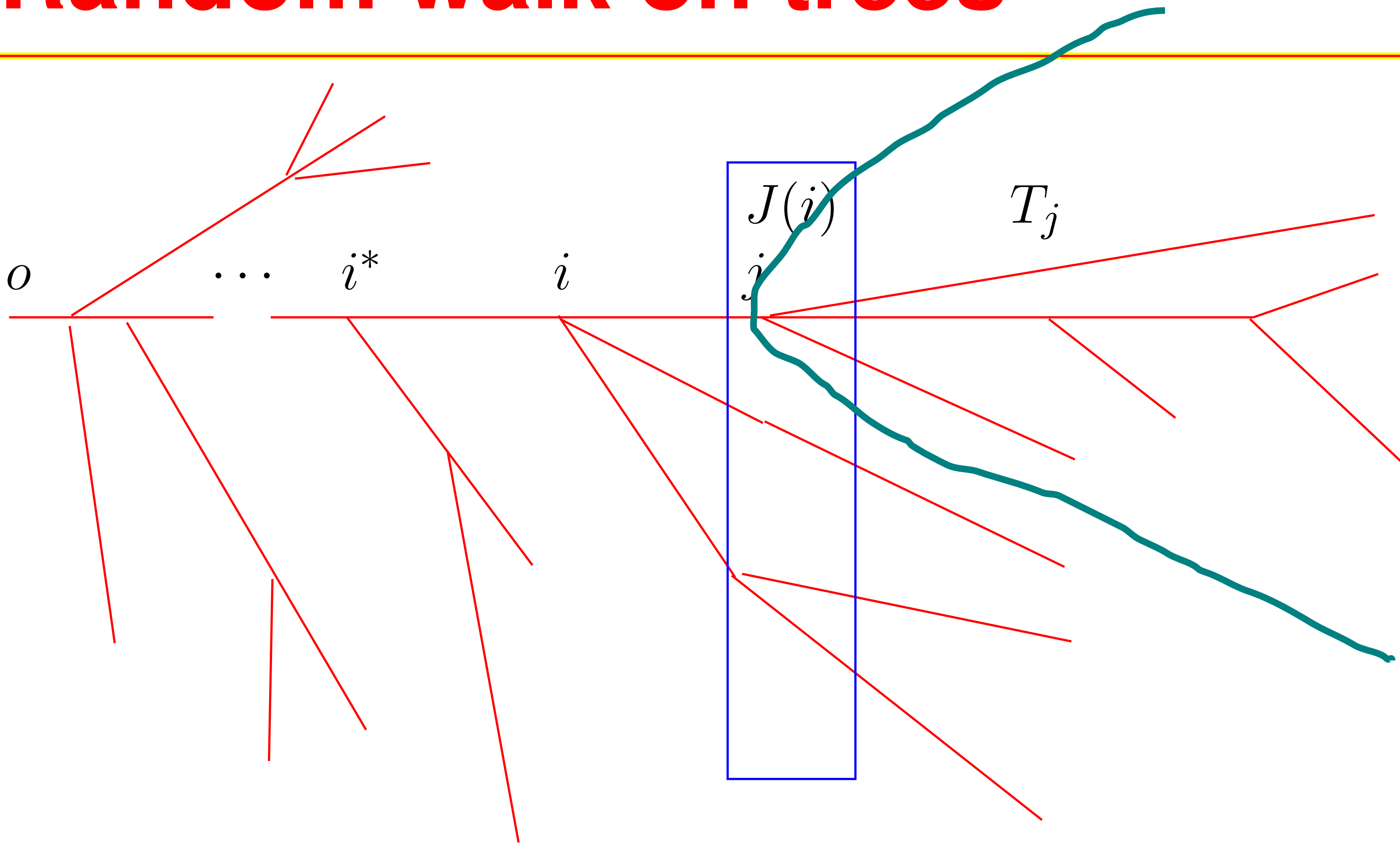
Random walk on trees

- time–continuous random walk on T
- Tree $T = (o, V, E)$: root o, vertices V, edges E.
- $i \in V$, denote by i^* the parent of i, by $J(i)$ all children of i.
  For any $i \in V$, there exists a unique path (no loop) in the tree connecting i to o, the set of the vertices on the path (excluding o) is denoted by $P(i)$, i.e. $P(i) = \{i, i^*, i^{**}, \cdots\}$.

Random walk on trees

- time–continuous random walk on T
- Tree $T = (o, V, E)$: root o, vertices V, edges E.
- $i \in V$, denote by i^* the parent of i, by $J(i)$ all children of i.
  For any $i \in V$, there exists a unique path (no loop) in the tree connecting i to o, the set of the vertices on the path (excluding o) is denoted by $P(i)$, i.e. $P(i) = \{i, i^*, i^{**}, \cdots\}$.
- The subtree rooted from i is denoted by T_i.

Random walk on trees



Random walk on trees

- Let $a_i, b_i, i \in V$ be two sequences of positive reals. $Q = \{q_{ij} : i, j \in V\}$:

$$q_{ij} = \begin{cases} a_i, & \text{if} \quad j = i^*, i \neq o; \\ b_j, & \text{if} \quad j \in J(i); \\ 0, & \text{for other } i \neq j. \end{cases}$$

Random walk on trees

- Let $a_i, b_i, i \in V$ be two sequences of positive reals. $Q = \{q_{ij} : i, j \in V\}$:

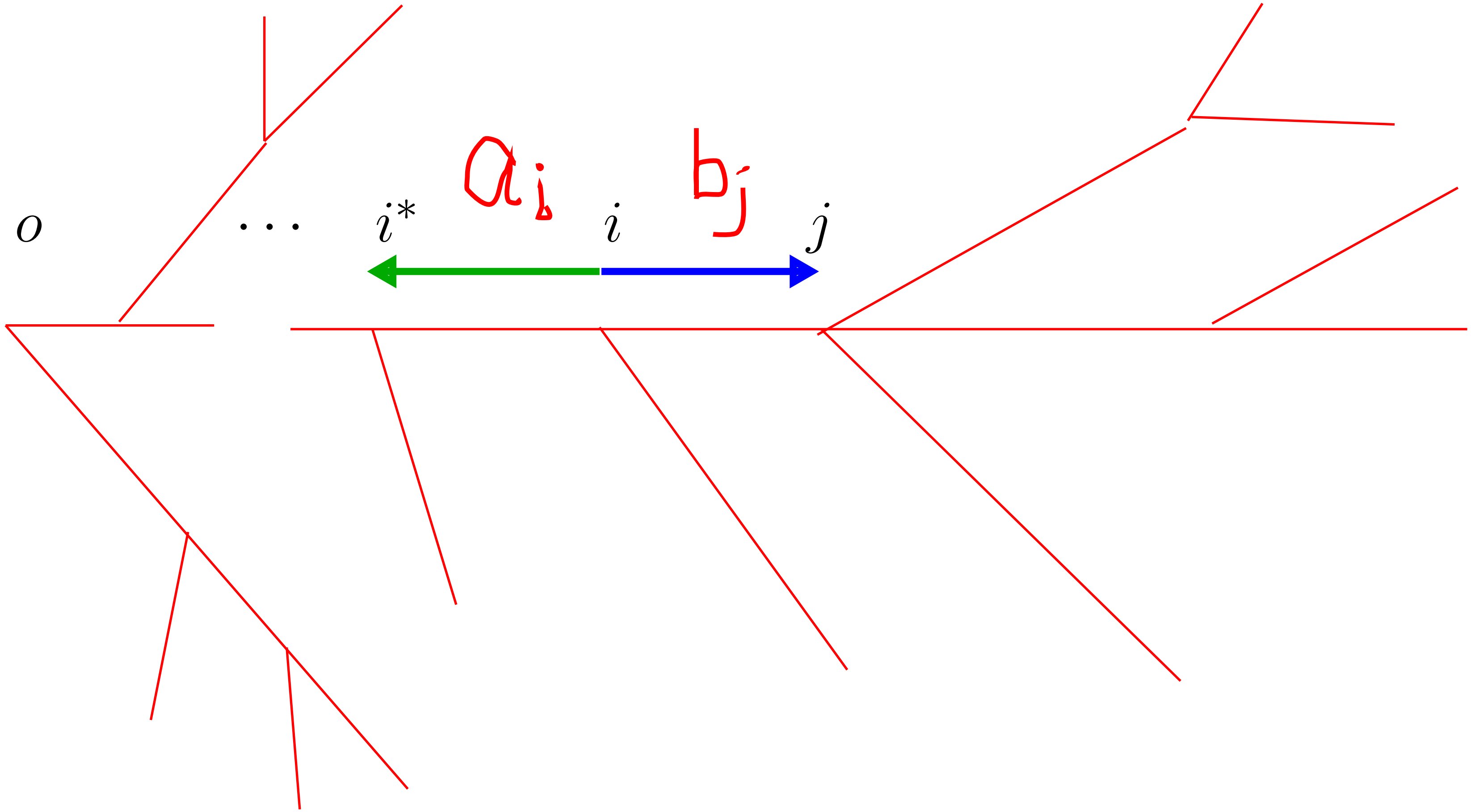
$$q_{ij} = \begin{cases} a_i, & \text{if} \quad j = i^*, i \neq o; \\ b_j, & \text{if} \quad j \in J(i); \\ 0, & \text{for other } i \neq j. \end{cases}$$

- reversible w.r.t. $\pi = (\pi_i, i \in V)$ defined by

$$\mu_o = 1, \mu_i = \prod_{j \in P(i)} \frac{b_j}{a_j} \qquad (2)$$

with $\mu := \sum_{i \in V} \mu_i < \infty$ so that $\pi_i = \mu_i / \mu$

Random walk on trees



Random walk on trees

- For regular random walk on tree, strong ergodicity iff

$$\sup_i \sum_{j\in P(i)} \frac{\mu(T_j)}{\mu_j a_j} < \infty$$

which is equivalent to $\sigma_{\text{ess}}(L|L^\infty) = \emptyset$

QED

Thank You !