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On the regularity of affine Markov processes

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Based on a part of the preprint “Skew convolution semigroups and affine Markov processes” (by Dawson and Li).

1. Stochastic interest rate models

Example 1 Cox-Ingersoll-Ross model ('85):

$$dx(t) = (b + \beta x(t))dt + \sigma\sqrt{2x(t)}dB(t). \tag{1}$$

Also known as a CBI diffusion; see Kawazu-Watanabe ('71).

Example 2 Vasicek model ('77):

$$dz(t) = (b + \beta z(t))dt + \sigma\sqrt{2}dB(t). \tag{2}$$

Also known as an OU diffusion.

Example 3 Affine model:

$$\begin{aligned} dx(t) &= (b_1 + \beta x(t))dt + \sigma_{11}\sqrt{2x(t)}dB_1(t) \quad (3) \\ dz(t) &= b_2 dt + \sigma_{21}\sqrt{2x(t)}dB_1(t) \\ &\quad + \sigma_{22}\sqrt{2x(t)}dB_2(t). \quad (4) \end{aligned}$$

An extension of the Black-Scholes model (geometric BM) to stochastic volatility; Heston ('93).

2. Regular affine Markov processes

Let $D = \mathbb{R}_+^m \times \mathbb{R}^n$. A Markov semigroup $(P_t)_{t\geq 0}$ on D is called affine if

$$\int_D \exp\{\langle u, \xi\rangle\} P_t(x, d\xi) = \exp\{\langle x, \psi(t,u)\rangle + \phi(t,u)\} \quad (5)$$

for all $u \in U$ (suitably chosen); Duffie et al (AAP '03, 984-1053).

Theorem 0 (Duffie et al, '03) If $(P_t)_{t\geq 0}$ is regular (i.e., $\psi_t'(0+,u)$ and $\phi_t'(0+,u)$ exist), then

$$\psi_t'(t,u) = R(\psi(t,u)), \quad \psi(0,u) = u \tag{6}$$

and

$$\phi(t,u) = \int_0^t F(\psi(s,u))ds, \tag{7}$$

where

$$\begin{aligned} F(u) &= b_1u_1 + b_2u_2 + au_2^2 \\ &\quad + \int_D (e^{\langle u,\xi\rangle} - 1 - \chi(\xi_2)u_2)m(d\xi) \end{aligned} \tag{8}$$

and $R(u)$ has a similar representation.

Motivation: Prove that all affine semigroups are regular.

3. A decomposition of affine semigroups

Let $D = \mathbb{R}_+ \times \mathbb{R}$. An affine semigroup $(Q(t))_{t\geq 0}$ is called homogeneous if

$$\int_D \exp\{\langle u, \xi\rangle\} Q(t, x, d\xi) = \exp\{\langle x, \psi(t, u)\rangle\}. \tag{9}$$

A family of probabilities $(\gamma(t))_{t\geq 0}$ on D is a skew convolution semigroup (SC-semigroup) if

$$\gamma(r + t) = \{\gamma(r)Q(t)\} * \gamma(t), \qquad r, t \geq 0. \tag{10}$$

Proposition 1 If $(\gamma(t))_{t\geq 0}$ is a SC-semigroup, then $P(t, x, \cdot) = Q(t, x, \cdot) * \gamma(t, \cdot)$ defines a general affine semigroup. Indeed,

$$\int_D \exp\{\langle u, \xi\rangle\} \gamma(t, d\xi) = \exp\{\phi(t, u)\}. \tag{11}$$

4. Regularity of SC-semigroups

Let $(Q(t))_{t\geq 0}$ be a homogeneous affine semigroup and $(\gamma(t))_{t\geq 0}$ an associated SC-semigroup, both stochastically continuous.

Proposition 2 We have (infinite divisibility)

$$\begin{aligned}\phi(t,u) &= b_1(t)u_1 + b_2(t)u_2 + a(t)u_2^2 \\ &\quad + \int_D (e^{\langle u,\xi\rangle} - 1 - \chi(\xi_2)u_2)m(t,d\xi) \end{aligned} \tag{12}$$

and (key relations)

$$b_1(r+t) = b_1(r)\beta_{11}(t) + b_1(t), \tag{13}$$

$$b_2(r+t) = b_1(r)\beta_{12}(t) + b_2(r)\beta_{22}(t) + b_2(t) + \int_D [Q(t)\chi_2(\xi) - \beta_{22}(t)\chi(\xi_2)]m(r, d\xi), \tag{14}$$

$$a(r+t) = b_1(r)\alpha(t) + a(r)\beta_{22}^2(t) + a(t), \tag{15}$$

$$m(r+t,\cdot) = \int_D m(r, d\xi)Q(t,\xi,\cdot) + b_1(r)\mu(t,\cdot) + m(t,\cdot), \tag{16}$$

where $\alpha(t)$, $\beta_{ij}(t)$ and $\mu(t, d\xi)$ are determined by $\psi(t, u)$.

Lemma 1 The function

$$t \mapsto b_1(t) + \int_D \chi(\xi_1) m(t, d\xi) \tag{17}$$

is absolutely continuous.

Let $0 \leq r_1 < t_1 < \cdots < r_n < t_n \leq T$ and

$$\sigma_n = \sum_{j=1}^{n} (t_j - r_j). \tag{18}$$

Lemma 2 We have

$$\sum_{j=1}^{n}[a(t_j) - a(r_j)] \leq C(T)[b_1(\sigma_n) + a(\sigma_n)]. \qquad (19)$$

Lemma 3 Set $g(t) = \int_D \chi^2(\xi_2) m(t, d\xi)$. Then

$$\sum_{j=1}^{n}[g(t_j) - g(r_j)] \leq C(T)[b_1(\sigma_n) + g(\sigma_n)]. \qquad (20)$$

(Roughly, continuity implies absolute continuity.)

Proof of Lemma 3 Form (), $g(t)$ is non-decreasing in $t \geq 0$ and

$$\begin{aligned} g(t_1) - g(r_1) &= \int_{\mathbb{R}} m(t_1 - r_1, d\xi) \int_{\mathbb{R}} \chi^2(\eta_2) Q_{r_1}(\xi, d\eta) \\ &\quad + b_1(t_1 - r_1) \int_{\mathbb{R}} \chi^2(\xi_2) \mu(r_1, d\xi) \\ &= \int_{\mathbb{R}} \chi^2(\beta_{22}(r_1)\xi_2) m(t_1 - r_1, d\xi) \\ &\quad + b_1(t_1 - r_1) \int_{\mathbb{R}} \chi^2(\xi_2) \mu(r_1, d\xi) \\ &\leq \int_{\mathbb{R}} \chi^2(C(T)\xi_2) m(t_1 - r_1, d\xi) \\ &\quad + b_1(t_1 - r_1) \int_{\mathbb{R}} \chi^2(\xi_2) \mu(r_1, d\xi) \\ &\leq C(T) \int_{\mathbb{R}} \chi^2(\xi_2) m(t_1 - r_1, d\xi) + C(T) b_1(t_1 - r_1) \\ &\leq C(T)[b_1(t_1 - r_1) + g(t_1 - r_1)]. \end{aligned}$$

That is, the result holds for $n = 1$. The general case follows by more careful analysis. □

Theorem 1 If $t \mapsto b_2(t)$ is absolutely continuous, $t \mapsto \phi(t, u)$ is differentiable.

Sketch of Proof (Absolute continuity implies differentiability.)

(1) By Lemmas 1 – 3,

$$\phi(t, u) = \int_0^t [\log \hat{\nu}_s(u)] ds \tag{21}$$

for a family of infinitely divisible probabilities $(\nu_s)_{s>0}$.

(2) By (), we modify the definition of $(\nu_s)_{s>0}$ to get an entrance law for $(Q(t))_{t\geq 0}$.

(3) The Feller property of $(Q(t))_{t\geq 0}$ implies that $(\nu_s)_{s>0}$ can be closed by some ν_0. Then $t \mapsto \phi(t, u)$ is differentiable. □

5. Regularities under moment conditions

Suppose that $(Q(t))_{t\geq 0}$ and $(\gamma(t))_{t\geq 0}$ are stochastically continuous.

Theorem 2 Suppose that

$$\int_D (\xi_1 + |\xi_2| \wedge |\xi_2|^2) m(t, d\xi) < \infty, \tag{22}$$

then $t \mapsto \phi(t, u)$ is differentiable.

(Roughly, the first moment condition implies differentiability.)

In particular, () holds if

$$\int_D (\xi_1 + |\xi_2|^2) \gamma(t, d\xi) < \infty. \tag{23}$$

Sketch of Proof of Theorem 2 Let $x(\cdot)$ be a Hunt realization.

(1) Find g_1 and g_2 in the domain of generator so that

$$\exp\{\langle u, x(t)\rangle\} = H_u(g_1(x(t)), g_2(x(t))) \tag{24}$$

for a smooth H_u. Then $\exp\{\langle u, x(t)\rangle\}$ is a semi-martingale.

(2) Prove that

$$t \mapsto \phi(t, u) = \log E_0 \exp\{\langle u, x(t)\rangle\} \tag{25}$$

is a.e. differentiable.

(3) From () infer that $b_2(t)$ is a.e. differentiable and hence continuously differentiable in $t \geq 0$. Then apply Theorem 1. □

6. Concluding remarks

The proofs rely heavily on the analysis of the relation

$$\gamma(r+t) = \{\gamma(r)Q(t)\} * \gamma(t), \qquad r, t \geq 0. \tag{26}$$

and its consequences. (Stochastic continuity $\longrightarrow$ absolute continuity $\longrightarrow$ differentiability.)

Applications to immigration superprocesses; Li (1995, 1996, 1998, 1999, 2002), Dawson-Li ('03), etc.

Applications to Ornstein-Uhlenbeck processes on Hilbert spaces; Bogachev et al (1996), Fuhrman-Röckner ('00), Dawson-Li ('04), Dawson et al ('04), etc.

Thanks!

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