Liouville theorem and Feller property for diffusion operator

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1. Introduction **Strong Liouville Theorem** Every non-negative harmonic function on $\mathbb{R}^n$ must be a constant.

$$\Delta u = 0,\ u \geq 0 \Longrightarrow u = Cte. \tag{1}$$

Weak Liouville Theorem Every bounded harmonic function on $\mathbb{R}^n$ must be a constant.

$$\Delta u = 0,\ u \in L^\infty \Longrightarrow u = Cte. \tag{2}$$

Feller property The set $C_0(\mathbb{R}^n)$ is stable under the heat semi-group $P_t = e^{t\Delta}$ on $\mathbb{R}^n$.

$$f \in C_0(\mathbb{R}) \Longrightarrow P_t f \in C_0(\mathbb{R}),\ \ \forall\, t > 0. \tag{3}$$

Here $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \lim_{|x|\to\infty} f(x) = 0\}.$

Proof I: Strong Liouville follows from elliptic Harnack inequality

Let u be any non-negative solution of $\Delta u = 0$ in a geodesic ball $B(o, R)$. Then the elliptic Harnack inequality (EHI) holds

$$\sup_{B(o,R/2)} u \leq C \inf_{B(o,R/2)} u,$$

where C is independent of u and $B(o, R)$. Let u be a solution of $\Delta u = 0$ in M which is bounded from below. Applying the EHI to $v = u - \inf_M u$ in a geodesic ball $B(o, 2R)$, we obtain

$$\sup_{B(o,R)} (u - \inf_M u) \leq C \inf_{B(o,R)} (u - \inf_M u).$$

Letting $R \to \infty$, the (RHS) tends to zero. Hence $u = \inf_M u$.

Elliptic and parabolic Harnack inequalities for uniformly elliptic operator

(PHI)$\Longrightarrow$ (EHI)$\Longrightarrow$ Strong Liouville$\Longrightarrow$ Weak Liouville

Consider a second order elliptic operator (in divergence form) on $\mathbb{R}^n$:

$$L = \sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j) + \sum_i b_i(x) \partial_i + c(x).$$

Assume that L is uniformly elliptic: $A(x) = (a_{ij}(x))$ is symmetric, and there exists a constant $\lambda \in (0,1)$ such that

$$\lambda I \leq A(x) \leq \lambda^{-1} I, \ \forall\, x \in \mathbb{R}^n.$$

Question Under what condition on A, b and c, (PHI), (EHI), Strong and Weak Liouville hold ?

Theorem (De Giorgi-Nash-Moser) Let $L = \sum_{i,j} \partial_i(a_{ij}(x)\partial_j)$ be a uniformly elliptic operator with some $\lambda > 0$. Then, for any $\delta \in (0,1)$, there exists a constant $C = C(n, \lambda, \delta)$ such that any positive solution u of $Lu = 0$ in a ball $B \subset \mathbb{R}^n$ satisfies the elliptic Harnack inequality

$$\sup_{\delta B} u \leq C \inf_{\delta B} u.$$

Theorem (see Gilbarg and Trudinger) Let $L = \sum_{i,j} \partial_i(a_{ij}(x)\partial_j) + \sum_i b_i(x)\partial_i + c(x)$. Suppose that there exists a constant $\lambda > 0$ such that

$$\lambda I \leq A(x), \ \ \|A(x)\|_{HS} \leq \Lambda^2,$$

$$\frac{|b(x)|^2}{\lambda^2} + \frac{|c(x)|}{\lambda} \leq \nu^2.$$

Let $u \in W^{1,2}(\Omega)$ be a nonnegative solution of $Lu = 0$ in a domain $\Omega \subset \mathbb{R}^n$. Then, for any ball $B(x, 4R) \subset \Omega$, there exists a constant $C = C(n, \Lambda/\lambda, \nu R)$ such that

$$\sup_{B(x,R)} u \leq C \inf_{B(x,R)} u.$$

Remark $\nu \neq 0$ cannot implies the strong Liouville.

In 1964, Moser proved the parabolic Harnack inequality (PHI) which is stronger than (EHI). In 1992, based on some earlier works of Bombieri-Giusti (72) and S.T. Yau (75), Grigor'yan and Saloff-Coste independently extended Moser's (PHI) to complete non-compact Riemannian manifolds.

Theorem (Grigor'yan 92, Saloff-Coste 92): Let M be a complete Riemannian manifold satisfying
(a) The doubling volume property holds on M, i.e., there exists a constant $A > 0$ such that

$$V(B(x, 2R)) \leq AV(B(x, R)), \;\; \forall\, x \in M, \;\; R > 0.$$

(b) The local Poincaré inequality holds on any geodesic ball, i.e., there exists a constant $a > 0$ such that for any $x \in M, R > 0$

and any $f \in C^1(B(x,R))$,

$$\int_{B(x,2R)} |\nabla f|^2 dx \geq \frac{a}{R^2} \inf_{c \in \mathbb{R}} \int_{B(x,R)} |f(x) - c|^2 dx.$$

Then there exists a constant $C > 0$ such that for any positive solution of $\partial_t u = \Delta u$, the parabolic Harnack inequality holds: for any $x \in M$ and $R > 0$,

$$\sup_{\mathcal{C}_1} u \leq C \inf_{\mathcal{C}_2} u,$$

where $\mathcal{C}_1 = B(x,R) \times [R^2, 2R^2]$, $\mathcal{C}_2 = B(x,R) \times [3R^2, 4R^2]$. Moreover, the converse is also true. That is,

$$(a) + (b) \Longleftrightarrow (PHI) \Longrightarrow (EHI)$$

Proof II: Using the mean value theorem and the gradient estimate

Let $u(x)$ be a harmonic function on $D \subset \mathbb{R}^n$. Then the mean value theorem holds

$$u(x) = \frac{1}{\omega_n R^n} \int_{\partial B(x,R)} u(y) d\sigma_{n-1}(y).$$

Since u is harmonic, $\frac{\partial u}{\partial x_i}$ is also harmonic. Thus

$$\frac{\partial u(x)}{\partial x_i} = \frac{1}{\omega_n R^n} \int_{\partial B(x,R)} \frac{\partial u(y)}{\partial x_i} d\sigma(y).$$

By divergence theorem, we can prove that

$$|\nabla u(x)| \leq \frac{n\omega_{n-1}}{(n-1)\omega_n d}(M - m), \tag{4}$$

where $d = dist(x, \partial D)$, $M = \sup_M u$, $m = \inf_M u$.

As pointed out in M. H. Protter and H.F. Weinberger (Maximum Principles in Differential Equations, Prentice-Hall, 1967), for a general type of second order uniformly elliptic operator

$$L = \sum_{i,j} \partial_i(a_{ij}(x)\partial_j) + \sum_i b_i(x)\partial_i + c(x),$$

the derivatives of the solution of $Lu = 0$ satisfy some gradient estimates which are very similar to (4). However, the method to establish these gradient estimates is not elementary, since the mean value property does not hold in general, and the partial derivatives $v = \partial_i u$ of u are not solutions of the same elliptic equation $Lv = 0$. See L. Bers, F. John and M. Schechter: Partial Differential Equations, New York, Interscience Publishers, Inc., 1964, C. Miranda: Equazioni alle derivate parziali di tipo ellittico, Berlin, Springer-Verlag OHG, 1955, Gilbarg and Trudinger's book.

2. Liouville theorem on complete Riemannian manifold

Let (M, g) be a complete non-compact Riemannian manifold with Riemannian metric g, Δ be the Laplace-Beltrami operator on M. In a local coordinate $x = (x_1, \ldots, x_n)$, for any $f \in C_c^2(M)$,

$$\Delta f(x) = \frac{1}{\sqrt{detg(x)}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij}(x) \sqrt{detg(x)} \frac{\partial f}{\partial x_j} \right). \quad (5)$$

Theorem (S.T. Yau, 75) (1) Let M be a complete non-compact Riemannian manifold. Then for any $1 < p < \infty$, every non-negative L^p-subharmonic function on M or every L^p-harmonic function must be a constant.

$$\Delta u = 0,\ u \in L^p(\nu) \Longrightarrow u = Cte, \tag{6}$$

where

$$\nu(dx) = \sqrt{detg(x)}dx \tag{7}$$

denotes the Riemannian volume measure on (M, g).

(2) Let (M, g) be a complete non-compact Riemannian manifold with non-negative Ricci curvature. Then the strong Liouville theorem holds.

$$\Delta u = 0,\ u \geq 0 \Longrightarrow u = Cte. \tag{8}$$

To prove this result, Yau first established a gradient estimate for the harmonic function u which is bounded from below on a complete Riemannian manifold with Ricci curvature bounded from below. Suppose that $Ric \geq -K$, where $K \geq 0$. Then

$$|\nabla u| \leq \sqrt{(n-1)K}(u - \inf_M u).$$

Thus, if $Ric \geq 0$, then every non-negative harmonic function must be a constant.

Theorem (S.T. Yau, 76) Let (M, g) be a complete non-compact Riemannian manifold with Ricci curvature bounded from below. Then every non-negative bounded solution to $(\Delta - \alpha)u = 0$ on (M, g) is identically zero.

$$(\Delta - \alpha)u = 0,\ u \in L^\infty \Longrightarrow u = 0. \qquad (9)$$

Remarks Equivalence between analytic and probabilistic characterisations
(see e.g. E.B. Davies, Grigor'yan, K.T. Sturm, Wu-Zhang)

(1)

$$\begin{aligned} & \textbf{strong Liouville for } Lu \leq 0 \\ \iff & \textbf{ weak Liouville for } Lu \geq 0 \\ \iff & \textbf{ recurrence of } L\textbf{--diffusion} \\ \Longrightarrow & \textbf{ strong Liouville for } Lu = 0. \end{aligned}$$

Question: What is the probabilistic characterisation of the strong (weak) Liouville theorem for harmonic function ?

strong Liouville $\iff$ Martin boundary is trivial

Martin, Doob, Hunt, Ancona, Kief, Murata weak Liouville $\Longleftrightarrow$ BM has succesful coupling

Linvall-Rogers, Kendall, Cranston, M.F. Chen, F.Y. Wang

(2) The L^∞-Liouville property of

$$(\Delta - \alpha)u = 0 \text{ for some or all } \alpha > 0,$$

i.e., every non-negative L^∞-(sub)solution of $(\Delta - \alpha)u = 0$ is identically zero, is equivalent to the stochastically completeness, which says that, for all $x \in M$, and all $t > 0$,

$$\int_M p_t(x, y) d\nu(y) = 1. \tag{10}$$

By Khas'minskii (60), these are equivalent to the uniqueness of the heat equation $\partial_t u = Lu$ with initial date $u(0) = f \in L^\infty$.

$$\begin{aligned} & \textbf{non explosion of } L \textbf{ -diffusion} \\ \iff & L^\infty - \textbf{Liouville}(\Delta - \alpha)u = 0 \textbf{ for all } \alpha > 0 \\ \iff & L^\infty - \textbf{Liouville}(\Delta - \alpha)u = 0 \textbf{ for some } \alpha > 0 \\ \iff & L^\infty - \textbf{uniqueness } \partial_t u = \Delta u. \end{aligned}$$

Theorem (Karp and P. Li, 83 unpublished) Let (M, g) be a complete Riemannian manifold on which

$$\nu(B(x, R)) \leq e^{CR^2}.$$

Then (M, g) is stochastically complete.

This result was also proved by different methods by E.B. Davies (J.d'Analyse Math. 92) and M. Takeda (Osaka J. Math. 89).

The optimal condition for the stochastical completeness is due to A.A. Grigor'yan.

Theorem (Grigor'yan, DAN SSSR 86, Soviet Math. Dokl. 87) Let (M, g) be a complete Riemannian manifold. Suppose that there exists a point $x \in M$ such that

$$\int_1^\infty \frac{r}{\log \nu(B(x,r))} dr = \infty.$$

Then (M, g) is stochastically complete.

Remarks (1) Let $K(r)$ be a lower bound of the Ricci curvature on the geodesic ball $B(x,r)$, i.e.,

$$Ric(x) \geq -K(r) \ \text{ on } \ B(x,r).$$

By the Bishop-Gromov volume comparison theorem, we have

$$\nu(B(x,r)) \leq Cr^n \exp\{\sqrt{(n-1)K(r)}r\}.$$

(2) Let (M, g) be a complete Riemannian manifold on which there exists a fixed point $o \in M$ such that the Ricci curvature on M

satisfies

$$Ric(x) \geq -C(1 + d^2(o, x)).$$

Then M is stochastically complete (P. Li, JDG, 86).

(3) Using a probabilistic approach, E. Hsu (Ann. Prob., 89) proved that (M, g) is stochastically complete provided that

$$\int_1^{\infty} \frac{dr}{\sqrt{K(r)}} = +\infty.$$

(4) The conditions given by the Ricci curvature are not stable under the quasi-isometry on the given manifold, while the conditions on the volume of geodesic ball are stable under quasi-isometry.

(5) (T. Lyons, JDG 87) The strong and the weak L^∞-Liouville properties as well as the stochastical completeness are not stable under the quasi-ismotry on complete non-compact Riemannian manifolds.

(6) Grigor'yan's criterion has been extended to a general setting of Dirichlet spaces by K.T. Sturm (J. Reine. Angew. Math. 94).

(7) Examples of complete non-compact Riemannian manifolds admiting non-constant non-negative and bounded harmonic functions: Cartan-Hadamard manifold with $-a \le \mathrm{Sect} \le -b$, where $a > b > 0$. See E. Dynkin (65), Prat (75), Anderson (82), Sullivan (82), Hsu and March (85), Hsu and Kendall (86), Anderson and Schoen (86), Ancona (87), Kief (87). See also S.Y. Cheng (92?), Ancona (00?), Hsu (03) for further results.

Question: What is the condition for the L^1-Liouville theorem ?

Theorem (1) (P. Li and R. Schoen, Acta Math. 84) Let (M, g) be a complete Riemannian manifold with

$$Ric(x) \geq -C(1 + r(x)^2)[\log(1 + r(x)^2]^{-\alpha}, \ \forall\, x \in M,$$

where $C > 0$ is a constant, $r(x) = d(o, M)$ is the distance function from a fixed point $o \in M$. Then M satisfies the L^1-Liouville property.

(2) (P. Li, JDG 84) The optimal condition for the L^1-Liouville property is given by

$$Ric(x) \geq -C(1 + r(x)^2), \ \forall\, x \in M.$$

Remarks

(1) Azencott (Bull. Soc. Math. Fr. 74) constructed complete two-diemnsional surfaces which have sectional curvature behaving like

$$K(x) \equiv -Cr^{2+\varepsilon}(x)$$

for some $C > 0$ and $\varepsilon > 0$. The Laplace-Beltrami operator on these surfaces has infinite many of fundamental solutions to the heat equation with the property that $p_t(x, y) \geq 0$ and

$$\int_M p_t(x, y)dy \leq 1.$$

Grigor'yan (87, 88) gave other counter-examples for the stochastical completeness.

(2) P. Li and Schoen (Acta Math. 84) and L.O. Chung (83) constructed counter-examples of complete non-compact Riemannian manifold on which the L^1-Liouville does not hold for $L = \Delta$.

(3) The non-negativity of the Ricci curvature (Yau 75) is not necessary for the L^∞-Liouville property.

(3.1) If (M, g) is quasi-isometric to a complete non-compact Riemannian manifold with nonnegative Ricci curvature, then Grigor'yan and Saloff-Coste's parabolic Harnack inequality (PHI) holds. This implies the strong and the weak Liouville theorem.

(3.2) A. Ancona (Revista Math Iberoamerica 00) constructed a CH manifold on which the Brownian motion converges to a single point in the sphere at infinity.

(3.3) F.-Y. Wang (SPA 02) found out a class of Liouville Riemannian manifolds with negative Ricci curvature

3. Feller property on complete Riemannian manifold

The heat semigroup $P_t = e^{t\Delta}$ on a complete non-compact Riemannian manifold is called to satisfy the Feller property (or the C_0-property) if

$$P_t(C_0(M)) \subset C_0(M),$$

where

$$C_0(M) = \{f \in C(M) : \lim_{r(o,x)\to\infty} f(x) = 0\}.$$

By Azencott (74), the Feller property holds iff for each $t > 0$ and for all compact subset $K \subset M$, it holds that

$$\lim_{x\to\infty} P_x(T_K < t) = 0, \tag{11}$$

where

$$T_K = \inf\{t > 0 : X_t \in K\}$$

is the entrance time of $\{X_t, t \geq 0\}$ in K.

Intuitively, P_t has the C_0-property iff the probability of the L-diffusion process starting from very far (near infinity) to visit any fixed compact subset before any fixed time is very small. Equivalently, P_t has C_0-property iff for $\lambda > 0$ the minimal positive solution of

$$(L - \lambda)u = 0 \text{ on } M \setminus K$$

with boundary condition $u \equiv 1$ on ∂K must tend to zero at infinity (the exterior Liouville property).

Azencott (74): C-H mainifold and homogenuous spaces Hsu-March (CPAM, 85): C-H manifold, Hsu (89): Ricci condition

Theorem (Hsu Ann.Prob. 89) Let M be a complete Riemannian manifold on which there exists a positive increasing continuous function $K(r)$ on $[0, \infty)$ such that

$$\inf\{Ric(x) : d(o,x) = r\} \geq -K(r)$$

and

$$\int_1^\infty \frac{dr}{\sqrt{K(r)}} = \infty.$$

Then the heat semigroup of the Laplace-Beltrami operator on M has the Feller proverty.

4. Case of diffusion operator on complete Riemannian manifold

Let (M, g) be a complete Riemannian manifold, $\phi \in C^1(M)$. Let L be a diffusion operator of the form

$$L = \Delta - \nabla\phi \cdot \nabla$$

which has an invariant measure given by

$$d\mu(x) = e^{-\phi(x)}\sqrt{detg(x)}dx.$$

The purpose of this talk is to study the following

Question: What is the natural and optimal condition on (M, g) and ϕ so that L has the L^∞, L^1-Liouville properties, the stochastical completeness and the Feller property ?

Remarks (1) Bakry (CRAS 84) proved that if

$$Ric(L) := Ric + \nabla^2 \phi$$

is uniformly bounded from below, then L is conservative. This is a natural extension of Yau's (75) result: If Ric is uniformly bounded from below, then (M, g) is stochastically complete.

(2) X.-M. Li (1993) extended Bakry's result to diffusion operator L satisfying

$$E \sup_{t \le T} \left[\exp - \frac{1}{2} \int_0^t Ric(L)(X_s) ds 1_{[t \le \xi(x)]} \right] < +\infty, \forall x \in U,$$

for some open set $U \subset M$ and some constant $T > 0$.

Question What happens when $Ric(L)$ is not uniformly bounded from below ?

Curvature-dimension condition $CD(K, m)$

Following Bakry, we call that L satisfies the curvature-dimension $CD(K, n)$ condition, if

$$\Gamma_2(u, u) \geq \frac{1}{n}(Lu)^2 + K|\nabla u|^2, \ \forall\, u \in C^\infty(M),$$

where

$$\Gamma_2(u, u) := \frac{1}{2}L|\nabla u|^2 - < \nabla Lu, u > .$$

By the classical Bochner-Weitzenböck formula, we can show that

$$\Gamma_2(u, u) = |\nabla^2 u|^2 + < Ric(L)\nabla u, \nabla u > .$$

Thus, $CD(K, n)$ holds when L is the Laplace-Beltrami operator Δ on a complete Riemannian manifold with $Ric \geq K$. Moreover, using the elementary inequality

$$(a+b)^2 \geq \frac{a^2}{1+\alpha} - \frac{b^2}{\alpha}, \ \forall\, \alpha > 0,$$

we obtain

$$\begin{aligned} G_2(u,u) &\geq \frac{1}{n}|\Delta u|^2 + < Ric(L)\nabla u, \nabla u > \\ &= \frac{1}{n}|Lu + \nabla\phi \cdot \nabla u|^2 + < Ric(L)\nabla u, \nabla u > \\ &\leq \frac{1}{n(1+\alpha)}|Lu|^2 - \frac{|\nabla\phi|^2|\nabla u|^2}{n\alpha} + < Ric(L)\nabla u, \nabla u > . \end{aligned}$$

Hence

$$\Gamma_2(u,u) \geq \frac{1}{n(1+\alpha)}|Lu|^2 + \left(Ric(L) - \frac{|\nabla\phi|^2}{n\alpha}\right)|\nabla u|^2.$$

Setting

$$m := (1+\alpha)n,$$

then

$$\Gamma_2(u,u) \geq \frac{1}{m}|Lu|^2 + \left(Ric(L) - \frac{|\nabla\phi|^2}{m-n}\right)|\nabla u|^2.$$

That is, the diffusion operator $L = \Delta - \nabla\phi\cdot\nabla$ on any n-dimensional complete Riemannian manifold (M,g) satisfies the $CD(K_{m,n}, m)$ condition for all $m > n$, where

$$K_{m,n} = Ric + \nabla^2\phi - \frac{|\nabla\phi|^2}{m-n}.$$

Bakry-Emery Ricci curvature via warped product metric

The Bakry-Emery Ricci curvature was introduced when Bakry and Emery (82) studied the logarithmic Sobolev inequality for diffusion operator on a complete Riemannian manifold. Since then it has played an important role in the study of functional inequality for symmetric Markovian semigroup. In recent papers by G. Perelman, it has been used in the proof of Poincar'e's conjecture on $\mathbf{3}$-manifold by modifying R. Hamilton's Ricci flow. In a recent paper, Lott gave a new understanding of the Bakry-Emery Ricci curvature by using the warped product metric.

Suppose that $q = m - n \in N$. Given $k \in Z^+$, consider $S^q \times M$

with the warped product metric defined by

$$g_k^{S^q\times M} = g^M + k^{-2}e^{-\frac{2\phi}{q}}g^{S^q}.$$

Equivalently, the Bakry-Emery Ricci curvature-dimension tensor satisfies

$$Ric_{n+q,n}^M(L)(X,X) = \pi_*\left(Ric^{S^q\times M}(\overline{X},\overline{X})\right).$$

Lott pointed out that, if $Ric_q \geq Kg$, then as $k\to\infty$, (M,g,ϕ) is the Hausdroff limit of a sequence of $m = n+q$-dimensional manifolds $(S^q\times M, g_k^{S^q\times M})$ with Ricci curvature bounded from below by K.

The main result of this talk is the following

Theorem (X.D. Li, 04) Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that $L = \Delta - \nabla\phi \cdot \nabla$ satisfies the following curvature-dimension condition: there exists a fixed point $o \in M$ and a constant $C \geq 0$ such that for all $x \in M$,

$$Ric(x) + \nabla^2\phi(x) - \frac{\nabla\phi(x) \otimes \nabla\phi(x)}{m-n} \geq -C(1 + r^2(x)). \tag{12}$$

Then (1) The strong and the weak L^∞-Liouville properties hold providing $C = 0$.

$$Lu = 0,\ u \geq 0 \implies u = Cte, \tag{13}$$

$$Lu = 0,\ u \in L^\infty \implies u = Cte. \tag{14}$$

(2) The stochastical completeness (i.e., the L^∞-Liouville property for $L - \alpha$, $\alpha > 0$) holds:

$$(L - \alpha)u = 0, \ u \in L^\infty \Longrightarrow u = 0. \tag{15}$$

(3) The L^1-Liouville property holds for L:

$$Lu = 0, \ u \in L^1(\mu) \Longrightarrow u = Cte. \tag{16}$$

where

$$\mu(dx) = e^{-\phi(x)}\sqrt{detg(x)}dx$$

denotes an invariant measure of L.

(4) The Feller property (i.e., the exterior Liouville property) holds:

$$f \in C_0(M) \Longrightarrow P_t f \in C_0(M).$$

5. The L^p-uniqueness of the heat equation

Strichartz (JFA 83) For $p \in (1, \infty)$, the Laplace-Beltrami operator generates a unique strongly continuous contractive L^p semigroup. Equivalently, for every $f \in L^p$, the heat equation $\partial_t u = \Delta u$ has a unique solution such that $u(0) = f$.

Theorem (X.D. Li 04) Suppose that

$$Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geq -C(1 + r(x)^2), \quad \forall\, x \in M.$$

(1) Let $v(x,t)$ be a non-negative function on $M \times \mathbb{R}^+$ such that

$$(L - \partial_t)\, v(x,t) \geq 0, \quad \int_M v(x,t)d\mu(x) < +\infty$$

for all $t > 0$, and

$$\lim_{t \to 0} \int_M v(x,t) d\mu(x) = 0.$$

Then $v(x,t) = 0$ for all $(x,t) \in M \times \mathbb{R}^+$.

(2) For any $f \in L^1(\mu)$, there exists a unique $L^1(\mu)$-solution to the heat equation $\partial_t u = Lu$ with the initial condition $u(0) = f$.

(3) For any $f \in C_b(M)$, there exists a unique bounded solution to the heat equation $\partial_t u = Lu$ with the initial condition $u(0) = f$.

6. Three Lemmas

Lemma 1 (Bakry-Qian 03) Suppose that

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n} \geq -K(r(x)).$$

Then

$$Lr(x) \leq a(r), \tag{17}$$

where $a(r)$ is the solution of the Riccati equation

$$-a'(r) = -K(r) + \frac{a^2(r)}{m-1}$$

with initial condition

$$\lim_{x\to 0} ra(r) = m-1.$$

Lemma 2 (Qian 97, Gong-Wang 01, Lott 03, Bakry-Qian 03) Suppose that there exists a constant $k > 0$ such that

$$Lr(x) \leq mk\coth[kr(x)], \quad \forall\, x \in M \setminus cut(o). \tag{18}$$

Then for all $r > 0$, and $\alpha > 1$,

$$\mu(B(o, \alpha r)) \leq \mu(B(o, r))\alpha^{m+1} e^{k\coth[mk(\alpha-1)r]}. \tag{19}$$

Lemma 3 (Kendall Ann.Prob 87, Cranston JFA 91) Let X_t be a L-diffusion process starting at $X_0 \in M$. Then there exist a standard one-dimensional Brownian motion β_t and a nondecreasing process L_t which is increasing only on $\{t : X_t \in cut(X_0)\}$ such that

$$r(x_t) = \beta_t + \frac{1}{2}\int_0^t Lr(X_s)ds - L_t.$$

7. Li-Yau type differential Harnack inequality

The Li-Yau differential Harnack inequality for the heat equation $\partial_t u = \Delta u$ has been improved by Bakry and Qian. Their technique and result remain valid for diffusion operator under curvature-dimension condition.

Theorem (Bakry-Qian Revista Mat. Iberoamer. 99) Suppose that $L = \Delta - \nabla\phi \cdot \nabla$ satisfies the curvature-dimension condition

$$Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geq -K$$

with $K \geq 0$. Let $f = \log u$, where u is a solution to the heat equation $\partial_t u = Lu$. Then

$$|\nabla f|^2 - f_t \leq \sqrt{mK}\sqrt{|\nabla f|^2 + \frac{m}{2t} + \frac{mK}{4} + \frac{m}{2t}}. \tag{20}$$

8. Sketch of Proof of the Main Theorem

(1) To prove the L^∞-Liouville property, we follow Yau's method to establish gradient estimate for the positive solution of $Lu = 0$.

(2) To prove the stochastical completeness, we verify that the Karp-Li condition holds:

$$\mu(B(x, r)) \le C_1 r^n e^{C_2 r^2}. \tag{21}$$

This follows from the Bishop-Gromov volume comparison theorem and the Bakry-Qian L-comparison theorem.

(3) To prove the Feller property, we follow the method used in Azencott, Hsu-March and Hsu. Using the Bakry-Qian L-comparison

theorem and the Kendall-Cranston formula, the following key estimate can be estabished.

Key Lemma Under the curvature-dimension condition, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that for all $n \geq 0$,

$$P_x\left(\theta_n \leq C/K(r(x)-n+1)\right) \leq e^{-C_2K(r(x)-n+1)}. \quad (22)$$

where $\theta_n = \tau_n - \sigma_n$, $\sigma_0 = 0$, and

$$\begin{aligned} \tau_n &= \inf\{t > \sigma_n : d(X_t, X_{\sigma_n}) = 1\}, \\ \sigma_n &= \inf\{t \geq \tau_{n-1} : d(o, X_t) = d(o,x) - n\}. \end{aligned}$$

That is, σ_n is the entrance time of the L-diffusion process X_t (starting at $x \in M$) in the geodesic ball $B(o, r(x) - n)$, $\theta_n := \tau_n - \sigma_n$ is the amount of time during which the L-diffusion process moves from $X_{\sigma_n} \in \partial B(o, r(x) - n)$ to $X_{\tau_n} \in \partial B(X_{\sigma_n}, 1)$, and $\sigma_{n+1} - \tau_n$ is the amount of time during which the L-diffusion process leaves from $\partial B(X_{\sigma_n}, 1)$ and hits $\partial B(o, r(x) - (n+1))$.

(4) The key steps in the proof of the L^1-Liouville property:
(4.1) for any nonnegative L^1-subharmonic function g, defining

$$e^{tL}g(x) := \int_M p_t(x,y)g(y)d\mu(y),$$

we need to prove the integration by parts formula

$$\int_M L_y p_t(x,y)g(y) = \int_M p_t(x,y)Lg(y)d\mu(y)$$

so that

$$\partial_t \left(e^{tL}g\right)(x) = \int_M p_t(x,y)Lg(y)d\mu(y).$$

For this, we need to prove that

$$\lim_{R\to\infty}\int_{\partial B(o,R)}|\nabla_y p_t(x,y)|g(y)d\mu(y)=0,$$

$$\lim_{R\to\infty}\int_{\partial B(o,R)}p_t(x,y)|\nabla g(y)|d\mu(y)=0.$$

(4.2) use the co-area formula and Bakry-Qian L-comparison theorem to prove

$$\sup_{B_o(R)} g(y)\leq Ce^{\alpha\sqrt{K(R)R}}\mu^{-1}(B_o(2R))\int_{B_o(2R)}g(y)d\mu(y)$$

for some constants $C,\alpha>0$ depending only on m., where $-K(R)$ is the lower bound of the Ricci curvature-dimension $Ric_{m,n}(L)$ on the geodesic ball $B_o(10R)$.

(4.3) use the Li-Yau differential Harnack inequality for ∂_t-L to prove the Li-Yau heat kernel upper bound under local $Ric_{m,n}(L)$ lower bound condition. That is, for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that for all $t > 0$, $x, y \in B_{x_0}(R)$, and some constant α depending only on m,

$$p_t(x,y) \leq \frac{C(\varepsilon)\exp\left(\frac{-d^2(x,y)}{(4+\varepsilon)t} + \alpha\varepsilon(K(R) + R^{-2})t\right)}{\sqrt{\mu(B_x(\sqrt{t})\mu(B_y(\sqrt{t})}}.$$

(4.4) prove the Cheng-Yau-Li type estimate for the heat kernel (need not to use the sectional curvature)

$$\left|\int_M |L_y p_t(x,y)|^2 d\mu(y)\right| \leq \frac{Cp_t(x,x)}{t^2}, \ \forall\, x \in M.$$

(4.5) using the conservativity result $\int_M p_t(x,y)d\mu(y) = 1$ to prove that every non-negative L^1-subharmonic function g must be harmonic. This is standard argument as used by P. Li. Then use the following result to conclude the L^1-Liouville theorem.

Theorem (Grigor'yan 88) Let L be a conservative diffusion operator, i.e., $\int_M p_t(x,y)dy = 1$, for all $x \in M$ and $t \geq 0$. Then every non-negative L^1-supsolution of $Lu \leq 0$ must be a constant.

Thanks!