随机延迟方程的数值解

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Numerical Solution of Stochastic Delay Equation

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Outline of The Talk

1. Numerical Solution of SDE
2. Stochastic Delay Equation
3. Numerical Scheme
4. Itô formula

1. Numerical Solution of SDE

Stochastic differential equation

$$dx(t) = b(x(t))dt + \sigma(x(t))dw_t$$

b and σ depend only on the current state $x(t)$.

Broad application

Explicit expression not available

Numerical Simulation (by computer)

Partition:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$$

On $t_k \leq t < t_{k+1}$

$$X_t = X_{t_k} + \int_{t_k}^{t} b(X_u)du + \int_{t_k}^{t} \sigma(X_u)dw_u$$

Approximately

$$X_t \approx X_{t_k} + b(X_{t_k})(t - t_k) + \sigma(X_{t_k})(w_t - w_{t_k})$$

Euler-Maruyama scheme

$$Y_{t_{k+1}} = Y_{t_k} + b(Y_{t_k})(t_{k+1} - t_k) + \sigma(Y_{t_k})(w_{t_{k+1}} - w_{t_k})$$

$$\tilde{X}_t = Y_{t_k} + \frac{Y_{t_{k+1}} - Y_{t_k}}{t_{k+1} - t_k}(t - t_{k+1})$$

$$\tilde{X}_t = \tilde{X}_{t_k} + b(\tilde{X}_{t_k})(t - t_k) + \sigma(\tilde{X}_{t_k})(w_t - w_{t_k})$$

Maruyama (1955) There is a constant C independent of partition π such that

$$\mathbf{E} \sup_{0 \le t \le T} |\tilde{X}_t - X_t|^2 \le C|\Delta|.$$

$\Delta = \max_{0 \le k \le n-1}(t_{k+1} - t_k)$

Order rate 0.5

Milstein Scheme

On $t_k \leq t < t_{k+1}$

$$X_t = X_{t_k} + \int_{t_k}^{t} b(X_u)du + \int_{t_k}^{t} \sigma(X_u)dw_u$$

Itô formula yields

$$\begin{aligned} \sigma(X_u) &\approx \sigma(X_{t_k}) + \int_{t_k}^{u} \sigma(X_v)\sigma'(X_v)dw_v + \cdots \\ &\approx \sigma(X_{t_k}) + \sigma(X_{t_k})\sigma'(X_{t_k})(w_u - w_{t_k}) + \cdots \end{aligned}$$

$$
\begin{aligned}
X_t &\approx X_{t_k} + b(X_{t_k})(t - t_k) + \int_{t_k}^{t} \Big[\sigma(X_{t_k}) \\
&\quad + \sigma(X_{t_k})\sigma'(X_{t_k})(w_u - w_{t_k})\Big] dw_u \\
&= X_{t_k} + b(X_{t_k})(t - t_k) + \sigma(X_{t_k})(w_t - w_{t_k}) \\
&\quad + \frac{1}{2}\sigma(X_{t_k})\sigma'(X_{t_k})(w_t - w_{t_k})^2
\end{aligned}
$$

Milstein scheme

$$\begin{aligned} Y_{t_{k+1}} &= Y_{t_k} + b(Y_{t_k})(t_{k+1} - t_k) \\ &\quad + \sigma(Y_{t_k})(w_{t_{k+1}} - w_{t_k}) \\ &\quad + \frac{1}{2}\sigma(Y_{t_k})\sigma'(Y_{t_k})(w_{t_{k+1}} - w_{t_k}) \end{aligned}$$

Theorem: C independent of π

$$\mathbf{E} \sup_{0 \le t \le T} |\tilde{X}_t - X_t|^2 \le C|\Delta|^2.$$

Rate (order) of convergence 1.0

Higher order schemes: What kind of terms should be included?

Rate Estimate?

Strong rate of convergence and Weak rate

References:

Kloeden P. and Platen, R., *Numerical Solution of Stochastic Differential Equations*, Springer, 1992.

Kloeden P., Platen, R., and Schurz, H., *Numerical Solution of SDE Through Computer Experiments*, Springer, 1994.

Hu, Y., *Strong and weak order of time discretization schemes of stochastic differential equations*, LNM **1626** (1996), 218-227.

2. Stochastic Delay Equation

Stochastic delay equation

b and σ may depend on the past.

Fix a number L.

Denote

$$x_t(s) := x(t+s)\,, \quad -L \le s \le 0\,.$$

$C = C([-L,0],\mathbf{R})$

Let b and σ be from $C([-L,0],\mathbf{R})$ to $\mathbf{R}$.

$$\begin{aligned} dx(t) &= b(x_t)dt + \sigma(x_t)dw(t) \\ &\qquad \text{if} \quad t \geq 0 \\ x(t) &= \phi(t) \\ &\qquad \text{if} \quad -L \leq t \leq 0 \end{aligned}$$

About the existence, uniqueness

Mohammed, S.-E. A., *Stochastic Functional Differential Equations*, Pitman Advanced Publishing Program, 1984.

Mohammed, S.-E. A., Stochastic differential systems with memory: Theory, examples and applications. Birkhauser (1998), 1-77.

Bell, D. and Mohammed, S.-E. A., *The Malliavin calculus and stochastic delay equations*, JFA **99** (1991), 75–99.

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Numerical Simulation?

Baker, C. T. H. and Buckwar, E., *Numerical analysis of explicit one-step methods for stochastic delay differential equations*., J. Comput. Math. **3** (2000), 315–335

Infinite dimensional stochastic differential equation.

Linear drift

We consider specific delay equations.

Define projection $\Pi : C \to \mathbf{R}^k$ associated with $s_1, \cdots, s_k \in [-L, 0]$

$$\Pi(\eta) := (\eta(s_1), \cdots, \eta(s_k)) \in \mathbf{R}^k$$

for all $\eta \in C$.

A function $\Phi \in C([0, T] \times C; R)$ is tame if

$$\Phi(t, \eta) = \phi(t, \Pi(\eta))$$

for all $t \in T$ and $\eta \in C$.

The stochastic delay equation that we consider is

$$\begin{aligned} X(t) &= \eta(0) + \int_0^t h(s, \Pi_2(X_s))ds \\ &\quad + \int_0^t g(s, \Pi_1(X_s))dw(s) \\ &\qquad \text{if } t \geq 0 \\ X(t) &= \eta(t) \qquad \text{if} - L \leq t < 0 \end{aligned}$$

where Π_1 and Π_2 are two projections associated with two sets of points

$$s_{1,1}, \cdots, s_{1,k_1} \in [-\tau, 0]$$

and

$$s_{2,1}, \cdots, s_{2,k_2} \in [-\tau, 0]$$

Assume

$$|g(t,x) - g(t,y)| \le L|x - y|,$$

$$|h(t,z) - h(t,w)| \le L|z - w|$$

$$\sup_{0 \le t \le a} \left[|g(t,0)| + |h(t,0)| \right] < \infty$$

Then the equation has a unique (strong) solution.

3. Numerical Schemes

Partition

$$\pi : 0 = t_0 < t_1 < t_2 < \cdots < t_n$$

$$|\pi| := \max_{0 \le i \le n-1} (t_{i+1} - t_i)$$

Euler-Maruyama scheme

$$\begin{aligned} X^\pi(t) &= X^\pi(t_i) + h(t_i, \Pi_2(X^\pi_{t_i}))(t - t_i) \\ &\quad + g(t_i, \Pi_1(X^\pi_{t_i}))(w(t) - w(t_i)) \end{aligned}$$

if $t \in [t_i, t_{i+1}]$

If $-r \leq t \leq 0$, then

$$X^\pi(t) = \eta(t)$$

where

$$X_t^\pi(s) = X^\pi(t+s),\ s \in [-L, 0], t \geq 0$$

Under some condition

$$E \sup_{0 \leq t \leq a} ||X_t^\pi - X_t||_C^q \leq C(q)|\pi|^{\frac{q}{2}}$$

for any $q \geq 1$.

Milstein Scheme

$$
\begin{aligned}
X^\pi(t) \;=\; & X^\pi(t_k) + h(t_k, \Pi_2(X^\pi_{t_k}))(t - t_k) \\
& + g(t_k, \Pi_1(X^\pi_{t_k}))(w(t) - w(t_k)) \\
& + \sum_{i=1}^{k_1} \frac{\partial g}{\partial x_i}(t_k, \Pi_1(X^\pi_{t_k}))u^\pi(t_k + s_{1,i}) \\
& I(t_k + s_{1,i}, t + s_{1,i}; s_{1,i})
\end{aligned}
$$

$$X^\pi(t) := \eta^\pi(t), \quad t \in [-L, 0]$$

$$\begin{aligned} u^\pi(t) &= g(t, \Pi_1(X^\pi_t)) \qquad \text{if} \quad t \geq 0 \\ u^\pi(t) &= \qquad \text{if} \quad -L \leq t < 0 \end{aligned}$$

$$\begin{aligned} & I\big(t_k + s_{1,i}, t + s_{1,i}; s_{1,i}\big) \\ = & \int_{t_k}^{t} \int_{t_k+s_{1,i}}^{t_1+s_{1,i}} \circ dw(t_2) \circ dw(t_1) \end{aligned}$$

Theorem: There is a constant C, independent of partition π

$$\mathbf{E} \sup_{0 \le t \le T} |X^\pi(t) - X(t)|^2 \le C|\pi|^2$$

4. Itô formula

$$
\begin{aligned}
dX(t) &= g(X(t-1), X(t))\, dw(t), \quad t \geq 0 \\
X(t) &= w(t), \quad -1 \leq t < 0.
\end{aligned}
$$

To find higher order scheme

$$\begin{aligned}
& d\{g(X(t-1), X(t))\} \\
= \; & d\{g(w(t-1), X(t))\} \\
= \; & \frac{\partial g}{\partial x}(w(t-1), X(t))\, dw(t-1) \\
& \quad + \frac{\partial g}{\partial y}(w(t-1), X(t)) \\
& g(X(t-1), X(t))\, dw(t) \\
& \quad + \text{second-order terms.}
\end{aligned}$$

Theorem Let

$$X(t) = \eta(0) + \int_0^t u(s)\, dw(s) + \int_0^t v(s)\, ds \quad \text{if} \quad t \geq 0$$

$$X(t) = \eta(t) \qquad \text{if} \quad -\tau \leq t \leq 0$$

$$
\begin{aligned}
& \phi(t, \Pi(X_t)) - \phi(0, \Pi(X_0)) \\
= \; & \int_0^t \frac{\partial \phi}{\partial s}(s, \Pi(X_s))\, ds \\
& + \int_0^t \frac{\partial \phi}{\partial \vec{x}}(s, \Pi(X_s))\, d(\Pi(X_s)) \\
& \quad + \frac{1}{2} \sum_{i,j=1}^{k} \sum_{i_1, j_1 = 1}^{m} \int_0^t \frac{\partial^2 \phi}{\partial x_{i_1 i} \partial x_{j_1 j}}(s, \Pi(X_s)) \\
& \quad u^{i_1}(s + s_i) D_{s+s_i} X^{j_1}(s + s_j)\, ds
\end{aligned}
$$

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Numerical Solution of Stochastic Differential Systems with Memory

Annals of Probability, 32 (2004), 265-314.

Thanks