Markov Skeleton Processes and Their Applications

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Abstract

In [1] and [2], we introduced the concept of Markov skeleton processes(MSP). Since then, we have begun to study MSP and their applications to queueing theory, risk theory, etc. Some remarkable results have been obtained ([3],[4]). For example, using MSP, we derive the transient distribution of the length of GI/G/N queueing systems, which was a open problem for almost 60 years.

This paper first gives the basic concepts of MSP and presents concise and strict treatments of the major results, then we discuss the limit behavior of MSP. In the following, we apply the major results in MSP to study queueing systems, inventory theory and reliability theory.

Outline

- Markov Skeleton Process
- Application to Queueing Systems
- Application to Inventory Theory
- Application to Reliability Theory

1. Markov skeleton processes

1.1. The Concept of Markov skeleton processes

Markov processes(MP): have the Markov property everywhere.

Markov skeleton processes(MSP) : have the Markov property at a series of stopping times(the only requirement).

$$\{MP\} \subset \{MSP\}$$

In the following, we give a strict definition of MSP.

$(E, \mathcal{E})$: a measurable space.

$(\Omega, \mathcal{F}, P)$: a complete probability space

$X = \{X(t, \omega), 0 \leq t < \infty\}$: a stochastic process defined on $(\Omega, \mathcal{F}, P)$ taking values in $(E, \mathcal{E})$

$\{\mathcal{F}_t^X, t \geq 0\}$: the natural flow of σ-algebras generated by X

θ_t:a shift operator, $(\theta_t w)_s = w_{t+s}, (w_s)_{s \geq 0} \in \Omega$.

Definition 1.1.1 *A stochastic process* $X = \{X(t, \omega), 0 \leq t < \infty)\}$ *is called a MSP if there exists a sequence of stopping times* $\{\tau_n\}_{n \geq 0}$*, satisfying*

(C1) $\tau_n \uparrow \infty$ *with* $\tau_0 = 0$*, and for each* $n \geq 0$*,* $\tau_n < \infty \Longrightarrow \tau_n < \tau_{n+1}$*;*

(C2) for all $n = 0, 1, \cdots$*, we have* $\tau_{n+1} = \tau_n + \theta_{\tau_n} \tau_1$*;*

(C3) For any $n > 0$*,* X *has the Markov property at* τ_n*,i.e. for every* τ_n

and any bounded $\mathcal{E}^{[0,\infty)}$*-measurable function* f *defined on* $E^{[0,\infty)}$

$$E[f(X(\tau_n + \cdot))|\mathcal{F}^X_{\tau_n}] = E[f(X(\tau_n + \cdot))|X(\tau_n)] \text{ P-a.s.}, \qquad (1.1.1)$$

where $\Omega_{\tau_n} = (\omega : \tau_n(\omega) < \infty)$, *and* $\mathcal{F}^X_{\tau_n} = \{A : \forall t \geq 0, A \cap (\omega : \tau_n \leq t) \in \mathcal{F}^X_t\}$ *is the* $\sigma-$*algebra on* Ω_{τ_n}. $\{\tau_n\}^{\infty}_{n=0}$ *is called skeleton time sequence of the MSP* X. *Furthermore, if on* Ω_{τ_n}

$$E[f(X(\tau_n + \cdot))|\mathcal{F}^X_{\tau_n}] = E[(f(X(\tau_n + \cdot))|X(\tau_n))]$$

$$= E_{X_{(\tau_n)}}[f(X(\cdot))](1.1.1')$$

P-a.s. holds, where $E_x(\cdot)$ *denotes the expectation corresponding to* $P(\cdot|X(0) = x)$. *Then we say that* X *is a time homogeneous MSP.*

Remark 1.1.1 *Here, suppose* E *to be a Polish space,* $\mathcal{E}$ *the Borel* σ-*algebra, and* Ω *be the right-continuous function space defined on* $\mathbb{R}^+(=[0,\infty))$ *with values in* E.

1.2. Backward equations

Let

$$q^{<n>}(x,t,A) = P(\tau_n \le t, X_{\tau_n} \in A | X(0) = x)$$

$$q(x,t,A) \triangleq q^{<1>}(x,t,A) = P(\tau_1 \le t, X_{\tau_1} \in A | X(0) = x)$$

$$q(x,ds,dy) = P(\tau_1 \in ds, X_{\tau_1} \in dy | X(0) = x)$$

Lemma 1.2.1 $q^{<n>}(x,t,dy)$,$x \in E, t \ge 0, A \in \varepsilon$ *satisfying the following conditions:*

(i) for fixed $A \in \varepsilon$,$q^{<n>}(\cdot,\cdot,A)$ *is a* $\varepsilon \times B(\mathbf{R}^+)$ *measurable function on* $E \times \mathbf{R}^+$*;*

(ii) for fixed $x \in E, t \ge 0$, $q^{<n>}(x,t,\cdot)$ *is a finite measure on* (E,ε)*;*

(iii) for any $t \geq 0, A \in \varepsilon$ *and* $m \in \mathbf{Z}^+ = \{0, 1, 2, \cdots\}$

$$q^{<n>}(X(\tau_m), t, A) = P\{X(\tau_{m+n}) \in A, \tau_{m+n} \leq t | X(\tau_m)\}, P - a.s.$$

Lemma 1.2.2 *For any* $n \in \mathbf{N}, t \geq 0, A \in \varepsilon, x \in E,$

$$\begin{aligned} q^{<n+1>}(x, t, A) &= \int_E \int_0^t q^{<n>}(x, ds, dy) q(x, t-s, A) \\ &= \int_E \int_0^t q(x, ds, dy) q^{<n>}(x, t-s, A) \end{aligned}$$

Definition 1.2.1 *A time homogeneous MSP* $X = \{X(t, \omega), \leq t < \infty\}$ *is called normal, if there exists a function* $h(x, t, A)$ *on* $E \times \mathbf{R}^+ \times \varepsilon$*, such that*

(i) for fixed x, t*,* $h(x, t, \cdot)$ *is a finite measure on* ε*;*

(ii) for fixed $A \in \varepsilon$,$h(\cdot,\cdot,A)$ *is* $\varepsilon \times \mathcal{B}(\mathbf{R}^+)$ *measurable function on* $E \times \mathbf{R}^+$*;*

(iii) for any $t \geq 0, A \in \varepsilon$*,*

$$h(X(\tau_n),t,A) = P\{X(\tau_n+t) \in A, \tau_{n+1}-\tau_n > t | X(\tau_n)\} \ P-a.s.$$

Let

$$P(x,t,A) = P\{X(t) \in A | X(0) = x\}$$

Theorem 1.2.1 *Suppose* $X = \{X(t); t \geq 0\}$ *is a normal MSP with* $\{\tau_n\}_{n=0}^{\infty}$ *as its skeleton time sequence, then for any* $x \in E, t \geq 0, A \in \varepsilon$*,*

$$P(x,t,A) = h(x,t,A) + \int_E \int_0^t \sum_{n=1}^{\infty} q^{<n>}(x,ds,dy) h(y,t-s,A) \quad (1.2.1)$$

thus $P(x,t,A)$ *is a minimal non-negative solution to the following non-negative equation system:*

$$P(x,t,A) = h(x,t,A) + \int_E \int_0^t q(x,ds,dy) P(y,t-s,A) \; x \in E, t \geq 0, A \in \varepsilon \tag{1.2.2}$$

Proof. For any $x \in E, t \geq 0, A \in \varepsilon, n \in \mathbf{N}$,

$$P(X(t) \in A, \tau_n \leq t < \tau_{n+1} | X(0) = x)$$

$$= \int_E \int_0^t P(X(\tau_n) \in A, \tau_n \leq t < \tau_{n+1} | X(\tau_n) = y, \tau_n = s, X(0) = x)$$

$$\cdot P(X(\tau_n) \in dy, \tau_n \in ds | X(0) = x)$$

$$= \int_E \int_0^t P(X(t-s+\tau_n) \in A, t-s < \theta_{\tau_n} \cdot \tau_1 | X(\tau_n) = y, \tau_n = s, X(0) = x)$$

$$q^{<n>}(x,ds,dy)$$

From $\tau_{n+1} = \tau_n + \theta_{\tau_n}\tau_1$, we know $\tau_{n+1} - \tau_n = \theta_{\tau_n}\tau_1$ is $\sigma(X(\tau_n + t); t \geq 0)-$measurable. And since that $X = \{X(t); t \geq 0\}$ is homogeneous on τ_n and $X = \{X(t); t \geq 0\}$ has the Markov property at τ_n, immediately we get

$$P(X(t - s + \tau_n) \in A, t - s < \theta_{\tau_n} \cdot \tau_1 | X(\tau_n) = y, \tau_n = s, X(0) = x)$$

$$= P(X(t - s) \in A, t - s < \tau_1 | X(0) = y) = h(y, t - s, A)$$

thus

$$P(X(t) \in A, \tau_n \leq t < \tau_{n+1} | X(0) = x) = \int_E \int_0^t q^{<n>}(x, ds, dy) h(y, t-s, A)$$

hence

$$P(x, t, A) = P(x(t) \in A | X(0) = x)$$

$$= P(X(t) \in A, t < \tau_1 | X(0) = x) + \sum_{n=1}^{\infty} P(X(t) \in A, \tau_n \leq t < \tau_{n+1} | X(0) = x)$$

$$= h(x,t,A) + \int_E \int_0^t \sum_{n=1}^{\infty} q^{<n>}(x,ds,dy)h(y,t-s,A)$$

i.e. $\{P(x,t,A); t \geq 0, x \in E\}$ satisfies equation (1.2.1). So $\{P(x,t,A)\}$ is the minimal non-negative solution to the equation system (1.2.2).

Equation system (1.2.2) is called backward equation system of the normal MSP$\{X(t), t \geq 0\}$.

Denote $\{h(x,t,A)\}$ by H, and $\{q(x,dt,dy)\}$ by Q. From Theorem 1.2.1, we know that one-dimensional distribution of $X(t)$ can be uniquely determined by (H,Q).

In our later discussion, MSP we may refer to are all normal.

Theorem 1.2.2

$\{X(t)\}$ *is a MSP* $+$ *its paths have left-hand limits with probability 1*

$\Longrightarrow \{X(t)\}$ *is normal.*

1.3. Limit distribution

Definition 1.3.1 *Suppose* $X(t)$*is a stochastic process on* $(\Omega, \mathcal{F}, \mathcal{P})$ *taking values in* $(E, \mathcal{E})$*.Let* $P(x,t,A) = P(X(t) \in A | X(0) = x)$*.If for any* $x \in E, A \in \mathcal{E}$*,* $\lim_{t\to\infty} P(x,t,A)$ *exists and is independent of* x*, and* $P(A) \equiv \lim_{t\to\infty} P(x,t,A) \quad (A \in \mathcal{E})$ *is the probability distribution on* $(E, \mathcal{E})$*, then we say that* **limit(probability) distribution** *of* $X(t)$ *exists, and* $P(\cdot)$ *is called the* **limit (probability) distribution** *of* $X(t)$*,*

Definition 1.3.2 *Suppose* $X(t)$ *is a MSP with* $\{\tau_n\}_{n=0}^{\infty}$ *as its skeleton time sequence. If there exists probability measure* $\pi(\cdot)$ *on* $(E, \mathcal{E})$*, such that* *for any* $A \in \mathcal{E}$,

$$P(X(\tau_1) \in A | X(0) = x, \tau_1 = s) = P(X(\tau_1) \in A) = \pi(A) \qquad (1.3.1)$$

then $X(t)$ *is called a* **Doob skeleton process**, $\pi(\cdot)$ *is called the* **characteristic measure** *of* $X(t)$ *and* $\tau_n, n = 1, 2, \cdots$ *is the* **regeneration point** *of* $X(t)$.

Remark 1.3.1 *The Doob skeleton process is a generalization of the Doob process in homogeneous denumerable MP. In applications, Doob skeleton processes or those obtained after choosing an appropriate skeleton time sequence are often encountered.*

Let

$$F(x,t) = P(\tau_1 \le t | X(0) = x), F(t) = \int_0^\infty \pi(dx) F(x,t)$$

Definition 1.3.3 *Suppose* $X(t)$ *is a Doob skeleton processes, if* $\int_0^\infty t dF(t) < \infty$ *and for any* $x \in E$, $F(x,0) = 0$, $F(x,\infty) \equiv 1$, *then* $X(t)$ *is called a* **positive recurrent** *Doob skeleton process.*

In the following, we give a sufficient condition for the existence of the limit distribution of a positive recurrent Doob skeleton process.

Theorem 1.3.1 *Suppose $X(t)$ is a positive recurrent Doob skeleton process. If $\pi(x: \quad F(x,\cdot)$ is absolutely continuous$) = 1$. then the stationary distribution $P(\cdot)$ of $X(t)$ exists, and we have*

$$P(A) = \frac{\int_0^\infty \int_E h(y,t,A)\Pi(dy)dt}{\int_0^\infty t dF(t)}, \quad (\forall A \in \mathcal{E})$$

1.4. Generalized limit distribution and invariable probability measure

Definition 1.4.1 *Suppose* $X(t)$ *is a stochastic process on* $(\Omega, \mathcal{F}, \mathcal{P})$ *taking values in* $(E, \mathcal{E})$*. If* $\Pi(\cdot)$ *is the probability measure on* $(E, \mathcal{E})$*, then for any* $A \in \mathcal{E}$*,* $x \in E$*,*

$$P(\lim_{t\to\infty} \frac{\int_0^t I_A(X(s))ds}{t} = \Pi(A)|X(0) = x) = 1,$$

we call $\Pi(\cdot)$ *is the* **generalized limit(probability) distribution** *of* $X(t)$*.*

Theorem 1.4.1 *Suppose* $X(t)$ *is a positive recurrent Doob skeleton process with* $\{\tau_n\}_{n=0}^{\infty}$ *as its skeleton time sequence, then generalized limit* *distribution* $\Pi(\cdot)$ *exists and*

$$\Pi(A) = \frac{\int_0^\infty \int_E h(y,t,A)\pi(dy)dt}{\int_0^\infty t dF(t)}, \quad \forall A \in \mathcal{E}.$$

Theorem 1.4.2 *Suppose* $\{X(t)\}$ *is a homogeneous MP taking values in* $(E, \mathcal{E})$*, and is a positive recurrent Doob skeleton process, then*

$$\Pi(A) = \frac{\int_0^\infty \int_E h(y,t,A)\pi(dy)dt}{\int_0^\infty t dF(t)}, \quad (A \in \mathcal{E})$$

is the unique invariable probability measure of $\{X(t)\}$*.*

Theorem 1.4.3 *Suppose* $X(t)$ *is a positive recurrent Doob skeleton process, if the limit distribution* $P(A)$ $(A \in \mathcal{E})$ *exists, then the generalized limit distribution* $\Pi(A)$ $(A \in \mathcal{E})$ *exists, and*

$$P(A) = \Pi(A) = \frac{\int_0^\infty \int_E h(y,t,A)\pi(dy)dt}{\int_0^\infty t dF(t)}$$

2. Queueing Systems

2.1. Introduction

$GI/G/1$ queueing system discussed here is as follows:

(i) the customers arrive in succession at times $\cdots \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \cdots$ and the interarrival times $t_m = \tau_m - \tau_{m-1} (m \in Z = (\cdots, -2, -1, 0, 1, 2, \cdots))$ are independent and uniformly distributed with distribution function $A(x)$

$$A(x) = P(t_m \leq x) \ (m \in Z) \tag{2.1.1}$$

(ii) the service times for each customer $\cdots \nu_{-3}, \nu_{-2}, \nu_{-1}, \nu_0, \nu_1, \nu_2, \cdots$ are independent of each other and so are all the $\{t_m, m \in Z\}$. And all ν_m are uniformly distributed with distribution function

$$B(x) = P(\nu_m \leq x) \ (m \in Z) \tag{2.1.2}$$

(iii)there is a service station which provides services according to the rule of FCFS(First Come, First Served)

Denote by $L(t)$ the length of $GI/G/1$ queuing system at time t (i.e. the total number of of waiting customers and those currently being served at time t).

Let $\theta_1(t)$ be the interval between time t and the arrival time of the last customer before time t.

$\theta_2(t)$ is defined as follows: if service station is idle at time t, then $\theta_2(t) = 0$, otherwise let $\theta_2(t)$ be the elapsed service time for the customer currently in service at time t.

It is well known that L(t) itself may not be a MP, but $(L(t), \theta_1(t), \theta_2(t))$ is a MP and also a MSP.

Let

$$A_{\theta_1}(t) = \frac{A(\theta_1 + t) - A(\theta_1)}{1 - A(\theta_1)} \tag{2.1.3}$$

$$B_{\theta_2}(t) = \frac{B(\theta_2 + t) - B(\theta_2)}{1 - B(\theta_2)} \tag{2.1.4}$$

Denote by $\mathcal{B}(I\!R^+)$ all the Borel sets on $I\!R^+ = [0, \infty)$, and $r_0 \equiv 0, r_1, r_2, \cdots$

be the discontinuous points of $(L(t),\theta_1(t),\theta_2(t))$(i.e., a customer arrives and/or a customer finishes the service and departs at time r_n).

$$r_n \uparrow +\infty, \quad (n \uparrow +\infty)\,. \tag{2.1.5}$$

Then $(L(t),\theta_1(t),\theta_2(t))$ is a MSP with $\{r_n\}$ as its skeleton time sequence.

For $i,j \in E=(0,1,2,\cdots),\theta_1,\theta_2 \in I\!R^+, A_1, A_2 \in \mathcal{B}(I\!R^+)$, let

$$h(i,\theta_1,\theta_2,j,A_1,A_2,t)$$

$$= P\left(L\left(t\right)=j,\theta_1\left(t\right)\in A_1,\theta_2\left(t\right)\in A_2, t<r_1 | L\left(0\right)=i,\theta_1\left(0\right)=\theta_1,\theta_2\left(0\right)=\theta_2\right)$$

$$q(i,\theta_1,\theta_2,ds,j,A_1,A_2)=$$

$$P\left(r_1\in ds, L\left(r_1\right)=j,\theta_1\left(r_1\right)\in A_1,\theta_2\left(r_1\right)\in A_2 | L\left(0\right)=i,\theta_1\left(0\right)=\theta_1,\theta_2\left(0\right)=\theta_2\right)$$

$$(2.1.6)$$

In particular, for $a, b \in I\!R^+$,

$$q(i, \theta_1, \theta_2, ds, j, a, b) \stackrel{\Delta}{=} q(i, \theta_1, \theta_2, ds, j, \{a\}, \{b\}) \qquad (2.1.7)$$

$$\begin{aligned} & P\left(i, \theta_1, \theta_2, j, A_1, A_2, t\right) \\ = \; & P\left(L\left(t\right) = j, \theta_1\left(t\right) \in A_1, \theta_2\left(t\right) \in A_2 | L\left(0\right) = i, \theta_1\left(0\right) = \theta_1, \theta_2\left(0\right) = \theta_2\right) \end{aligned}$$

$$(2.1.8)$$

Lemma 2.1.1

$$\begin{aligned} & h(i, \theta_1, \theta_2, j, A_1, A_2, t) \\ = & \begin{cases} 0, & j \neq i \\ I_{A_1}(\theta_1 + t)(1 - A_{\theta_1}(t)) I_{A_2}(0) I_{\{0\}}(\theta_2), & j = i = 0 \\ I_{A_1}(\theta_1 + t)(1 - A_{\theta_1}(t)) I_{A_2}(\theta_2 + t)(1 - B_{\theta_2}(t)) & j = i > 0, \end{cases} \end{aligned} \qquad (2.1.9)$$

Lemma 2.1.2

$$q\left(i,\theta_1,\theta_2,ds,j,A_1,A_2\right)$$
$$= \begin{cases} dA_{\theta_1}(s), & i=0, j=1, \theta_2=0, 0\in A_1, 0\in A_2 \\ (1-B_{\theta_2}(s))dA_{\theta_1}(s), & i\neq 0, j=i+1, 0\in A_1, s+\theta_2\in A_2 \\ (1-A_{\theta_1}(s))dB_{\theta_2}(s), & i\neq 0, j=i-1, s+\theta_1\in A_1, 0\in A_2 \\ (A_{\theta_1}(s)-A_{\theta_1}(s-))(B_{\theta_2}(s)-B_{\theta_2}(s-)), & j=i\neq 0, 0\in A_1, 0\in A_2 \\ 0, & \textit{otherwise} \end{cases} \tag{2.1.10}$$

In particular

$$\begin{cases} q(0,\theta_1,0,ds,1,0,0)=dA_{\theta_1}(s) \\ q(i,\theta_1,\theta_2,ds,i+1,0,s+\theta_2)=(1-B_{\theta_2}(s))dA_{\theta_1}(s), i\neq 0 \\ q(i,\theta_1,\theta_2,ds,i-1,s+\theta_1,0)=(1-A_{\theta_1}(s))dB_{\theta_2}(s), i\neq 0 \\ q(i,\theta_1,\theta_2,ds,i,0,0)=(A_{\theta_1}(s)-A_{\theta_1}(s-))(B_{\theta_2}(s)-B_{\theta_2}(s-)), i\neq 0 \end{cases} \tag{2.1.11}$$

Theorem 2.1.1 $\{P(i,\theta_1,\theta_2,j,A_1,A_2,t)\}$ *is the minimal nonnegative solution and also the unique bounded solution to the following nonnegative linear equation*

$$\begin{aligned}
& P(i,\theta_1,\theta_2,j,A_1,A_2,t) \\
= \; & h\left(i,\theta_1,\theta_2,j,A_1,A_2,t\right) \\
& + I_{\{0\}}(i)\int_0^t dA_{\theta_1}(s)P(i+1,0,\theta_2+s,j,A_1,A_2,t-s) \\
& + I_{\{1,2,\cdots\}}(i)(\int_0^t \left(1-A_{\theta_1}\left(s\right)\right)dB_{\theta_2}\left(s\right)P\left(i-1,\theta_1+s,0,j,A_1,A_2,t-s\right) \\
& + \sum_{s\le t}(A_{\theta_1}(s)-A_{\theta_1}(s-))(B_{\theta_2}(s)-B_{\theta_2}(s-))P(i,0,0,j,A_1,A_2,t-s) \\
& + \int_0^t (1-B_{\theta_2}(s))dA_{\theta_1}(s)P(i+1,0,\theta_2+s,j,A_1,A_2,t-s)) \qquad (2.1.12)
\end{aligned}$$

Let

$$d = \begin{cases} \inf\{t|L(t)=0\}, & \text{if } \{t|L(t)=0\} \text{ is a non-empty set,} \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$\gamma_0 = 0$,

$$\gamma_1 = \begin{cases} \inf\{t>d|L(t)=1\}, & \text{if } \{t>d|L(t)=1\} \text{ is a non-empty set,} \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\gamma_{n+1} = \gamma_n + \theta_{\gamma_n} \circ \gamma_1, (n = 1, 2, \cdots) \tag{2.1.13}$$

$$\hat{r}_0 = 0, \qquad \hat{r}_n = r_n \wedge \gamma_1, (n = 1, 2, \cdots) \tag{2.1.14}$$

Define

$$\hat{h}\left(i,\theta_1,\theta_2,j,A_1,A_2,t\right)=$$

$$P\left(L\left(t\right)=j,\theta_1\left(t\right)\in A_1,\theta_2\left(t\right)\in A_2,t<\hat{r}_1|L\left(0\right)=i,\theta_1\left(0\right)=\theta_1,\theta_2\left(0\right)=\theta_2\right) \tag{2.1.15}$$

$$\hat{q}\left(i,\theta_1,\theta_2,ds,j,A_1,A_2\right)=$$

$$P\left(\hat{r}_1\in ds,L\left(\hat{r}_1\right)=j,\theta_1\left(\hat{r}_1\right)\in A_1,\theta_2\left(\hat{r}_1\right)\in A_2|L\left(0\right)=i,\theta_1\left(0\right)=\theta_1,\theta_2\left(0\right)=\theta_2\right) \tag{2.1.16}$$

$$\hat{P}\left(i,\theta_1,\theta_2,j,A_1,A_2,t\right)=$$

$$P\left(L\left(t\right)=j,\theta_1\left(t\right)\in A_1,\theta_2\left(t\right)\in A_2,t<\gamma_1|L\left(0\right)=i,\theta_1\left(0\right)=\theta_1,\theta_2\left(0\right)=\theta_2\right) \tag{2.1.17}$$

then we have

$$\hat{h}\left(i,\theta_1,\theta_2,j,A_1,A_2,t\right)=h\left(i,\theta_1,\theta_2,j,A_1,A_2,t\right). \quad (2.1.18)$$

$$\begin{aligned}&\hat{q}\left(i,\theta_1,\theta_2,ds,j,A_1,A_2\right)\\ =&\begin{cases}0, & i=0\\ q(i,\theta_1,\theta_2,ds,j,A_1,A_2), & i>0.\end{cases}\end{aligned} \quad (2.1.19)$$

$$\hat{P}\left(0,\theta_1,\theta_2,j,A_1,A_2,t\right)=h\left(0,\theta_1,\theta_2,j,A_1,A_2,t\right) \quad (2.1.20)$$

Theorem 2.1.2 $\{\hat{h}(i,\theta_1,\theta_2,j,A_1,A_2,t)\}$ *is the minimal nonnegative solution to the following nonnegative linear equation.*

$$\hat{P}\left(0,\theta_1,\theta_2,j,A_1,A_2,t\right)=\hat{h}\left(0,\theta_1,\theta_2,j,A_1,A_2,t\right)$$

$$\hat{P}\left(i,\theta_1,\theta_2,j,A_1,A_2,t\right)$$

$$
\begin{aligned}
= \ & \hat{h}\,(i, \theta_1, \theta_2, j, A_1, A_2, t) \\
& + I_{\{1,2,\cdots\}}(i) \int_0^t (1 - A_{\theta_1}(s)) dB_{\theta_2}(s) \hat{P}(i-1, \theta_1 + s, 0, j, A_1, A_2, t-s) \\
& + \sum_{s \le t} (A_{\theta_1}(s) - A_{\theta_1}(s-))(B_{\theta_2}(s) - B_{\theta_2}(s-)) P(i, 0, 0, j, A_1, A_2, t-s) \\
& + \int_0^t (1 - B_{\theta_2}(s)) dA_{\theta_1}(s) \hat{P}(i+1, 0, \theta_2 + s, j, A_1, A_2, t-s). \\
& (i = 1, 2, \cdots, j = 0, 1, 2, \cdots, 0 \le \theta_1, \theta_2, t < \infty, A_1, A_2 \in \mathcal{B}(I\!R^+)
\end{aligned}
\tag{2.1.21}
$$

2.2. Statistical Equilibrium Theory (queue length)

In this section, we assume

$$
\begin{aligned}
0 < \frac{1}{\lambda} &= \int_0^\infty t dA(t) < \infty, \\
0 < \frac{1}{\mu} &= \int_0^\infty t dB(t) < \infty,
\end{aligned}
$$

Let $\rho = \frac{\lambda}{\mu}$.

Theorem 2.2.1 *If* $\rho < 1$ *then the generalized limit distribution* $\Pi(\cdot)$ *of* $(L(t), \theta_1(t), \theta_2(t))$ *exists:*

$$\Pi(j, A_1, A_2) = \frac{\int_0^\infty \hat{P}(1, 0, 0, j, A_1, A_2, t)dt}{\frac{1}{\lambda}\exp\{\sum_{k=1}^\infty \frac{1-a_k}{k}\}} \tag{2.2.1}$$

In particular

$$\Pi(j) = \frac{\int_0^\infty \hat{P}(1,0,0,j,[0,\infty),[0,\infty),t)dt}{\frac{1}{\lambda}\exp\{\sum_{k=1}^\infty \frac{1-a_k}{k}\}} \tag{2.2.2}$$

where $a_k = \int_0^\infty (1 - A^{<k>}(t))dB^{<k>}(t)$, *and* $A^{<k>}(t)$, $B^{<k>}(t)$ *denote convolutions of order* k *of* $A(t)$, $B(t)$ *respectively.*

Theorem 2.2.2 *If* $\rho < 1$ *and that* $A(t)$ *is absolutely continuous, then the limit distribution of* $(L(t), \theta_1(t), \theta_2(t))$ *exists and is equal to its generalized limit distribution.*

Theorem 2.2.3 *The necessary and sufficient condition for the existence of invariable measure of* $(L(t), \theta_1(t), \theta_2(t))$ *is that there exists a generalized limit distribution and they are equal when exist.*

2.3. Statistical Equilibrium Theory (waiting time)

Let $W(t)$ be the waiting time(including the service time) of the customer who arrives at time t in $GI/G/1$ queueing systems. And let $\theta(t) = \theta_1(t)$, obviously $(W(t), \theta(t))$ is a MP and also a MSP.

Using similar method in treating queue length, we can obtain some results.

3. Inventory Theory

3.1. Perishable inventory model

Perishable inventory can often be found in such fields as putrescible food inventory in supermarkets, easily-expired medicines in hospitals and weapons management in armies.

The above category of problems can be generalized as follows:

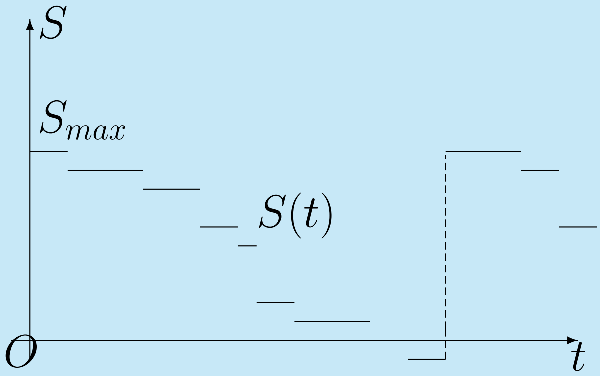
(1) Assume lifetimes of inventory commodities are i.i.d random variables,

with a common distribution function $F(t)$.

(2) Sell one commodity each time and the sale times of each commodity are i.i.d random variables, with a common distribution function $G(t)$. The sale times are also independent of the commodities'lifetimes.

(3) The maximum capacity of the warehouse is a fixed value S_{max} . When the capacity of the warehouse becomes $s(s < 0)$ (i.e. the quantity of OOS(Out Of Stock) arrives $-s$), buy in new commodities to increase the stocks until it reaches S_{max}.

$S(t)$ denotes the amount of stocks at time t. The path of $\{S(t); t \geq 0\}$ is shown in the following graph.



Generally speaking, $\{S(t); t \geq 0\}$ is not a MP.

Denote the lifetime of the commodity in stock at time t by $\theta(t)$, the time interval between the last sale before t and time t by $\hat{\theta}(t)$. Obviously $(S(t), \theta(t), \hat{\theta}(t))$ is a MP. Let $\tau_0 = 0$, τ_n be the $n-$th discontinuous point of $(S(t), \theta(t), \hat{\theta}(t))$ on $[0, \infty)$. Evidently $(S(t), \theta(t), \hat{\theta}(t))$ is a MSP with $\{\tau_n\}_{n=0}^{\infty}$ as its skeleton time sequence , therefore we can determine the transient distribution of $(S(t), \theta(t), \hat{\theta}(t))$ by the backward equations of the MSP.

Let $T_0 = 0$, and T_n denote the $n-$th time when $(S(t), \theta(t), \hat{\theta}(t))$ returns to the state of $(S_{max}, 0, 0)$, i.e. T_n denotes the $n-$th stocking time after the initial time 0. Obviously, $(S(t), \theta(t), \hat{\theta}(t))$ is a Doob skeleton process with skeleton time sequence $\{T_n\}_{n=0}^{\infty}$.

Let

$$
\begin{aligned}
h_j(t) &= P\{S(t) = j, t \leq T_1 | S(0) = S_{max}, \theta(0) = \hat{\theta}(0) = 0\}, \\
M &= E\{T_1 | S(0) = S_{max}, \theta(0) = \hat{\theta}(0) = 0\}.
\end{aligned}
$$

Following the method in proving Theorem 2.1.2, $h_j(t)$ can be uniquely determined by the minimal nonnegative solution to the nonnegative equation system. While

$$M = \int_0^\infty \sum_{i=s}^{S_{max}} h_j(t)dt.$$

From Theorems 1.3.3,1.4.1, 1.4.2 and 1.4.3, immediately we get:

Theorem 3.1.1 *(1)*$P\{T_1 < \infty | S(0) = S_{max}, \theta(0) = \hat{\theta}(0) = 0\} = 1$.

(2)if and only if $\int_0^\infty tF(dt) < \infty$ *or* $\int_0^\infty tG(dt) < \infty$, $M < \infty$. *Here the generalized limit distribution* $\Pi(\cdot)$ *of* $(S(t), \theta(t), \hat{\theta}(t))$ *exists, in particular*

$$\Pi(j) = \frac{\int_0^\infty h_j(t)dt}{M},$$

and $\Pi(\cdot)$ *is the unique invariable probability measure of* $(S(t), \theta(t), \hat{\theta}(t))$.

(3)if $M < \infty$ *and* $G(t)$ *is absolutely continuous, then the limit distribu-*

tion of $(S(t), \theta(t), \hat{\theta}(t))$ *exists and equals limit distribution.*

3.2. Reservoir Storage Model

In real reservoirs storage, the water drainage depends on the ingoing water speed and the water reserves in general, therefore a more practical reservoir model can be characterized as follows:

(1) Assume there is a reservoir with a capacity of $\bar{V} = Md_0$(where $d_0 > 0, M \in I\!N$). When the capacity of the reservoir reaches $\bar{V}$, the water in the reservoir will overflow automatically. $md_0, m = 1, 2, \cdots, M$ is called the water mark level. Denote the water reserves of the reservoir at time t by $V(t)$;

(2) There are N kinds of ingoing water speed $C_1, \cdots, C_N$. The water speed is controlled by a semi-Markov process $X(t)$ taking values in $E = \{1, 2, \cdots, N\}$, and the water speed at time t is $C_{X(t)}$;

(3) The outgoing water speed depends on the water reserves and the incoming water speed. To be concrete, when the water reserves is $v \in [md_0, (m+1)d_0)$, and the incoming water speed is C_i, the outgoing water speed is $\bar{C}(v, C_i) = \varphi(m, i)$, where $\varphi(m, i)$is a function of $m \in \{1, 2, \cdots, M\}$ and $i \in E$.

(4)Assume for any $m \in \{0, 1, 2, \cdots, M\}$, there exists $i, j \in E$, such that

$$\varphi(m, i) > C_i, \tag{3.2.1}$$

$$\varphi(m,j) \ < \ C_j. \tag{3.2.2}$$

For any $i \in E$,

$$\varphi(M,i) \ > \ C_i, \tag{3.2.3}$$

$$\varphi(0,i) \ < \ C_i. \tag{3.2.4}$$

First, we shall introduce some notations.

For any $v \in [0, \bar{V}]$ if $v \in [md_0, (m+1)d_0), m = 0, 1, \cdots, N$, then let $[v] = md_0$.

For any $v \in [0, \bar{V}], i \in E$, let

$$S(v,i) \ = \ \begin{cases} \frac{v-[v]}{\bar{C}(v,C_i)-C_i}, & \text{if } \bar{C}(v,C_i) > C_i; \\ \frac{[v]+d_0-v}{C_i-\bar{C}(v,C_i)} & \text{if } \bar{C}(v,C_i) < C_i. \end{cases}$$

$S(v,i)$ denotes the time needed for the water reserves v to reach the next water mark level when the water reserves is v , the incoming water speed is C_i, and the outgoing speed is $\bar{C}(v, C_i)$.

Let $\theta(t) = \inf\{s \geq 0, X(t-s) \neq X(t)\}$, and $\theta(t)$ denotes the time interval between the last discontinuous point $X(\cdot)$ before t and time t. Let $\tau_0 = 0$, and τ_n is the $n-$th discontinuous point of $(\bar{C}(V(t), C_{X(t)}), X(t))$ on $[0, \infty)$. Generally speaking, the water reserves process $\{V(t); t \geq 0\}$ is not a MP, while $\{(V(t), X(t), \theta(t))\}$ is a MP,and also a MSP with $\{\tau_n\}_{n=0}^{\infty}$ as its skeleton time sequence, thus we can determine the transient distribution of $\{(V(t), X(t), \theta(t))\}$ by the backward equations of MSP.

For this model, we can add some restrictions, for instance, the reservoir

$(V=0)$ can dry out with probability 1 , then $(V(t), X(t), \theta(t))$ is a Doob skeleton process and then we can judge its limit behavior. We will not discuss this problem here.

4. Reliability Theory

4.3. Parallel System

Suppose the system is composed of two different components and a repairman. The system is in good work state if two components work or one works and the other fails. The system is in failure state only if the two components fail. If a component breaks down, the repairman will repair the failed components immediately and when the failed component is repaired, it will begin to work at once. If the failed component is still in repairing process and the other component breaks down and in a to-be-repair state, we call the system in failure state.

Suppose the two components can be restored to the original state after each failure, the distribution function of the lifetime X_i of the $i-th$ component is

$$F_{X_i}(t) = P\{X_i \leq t\} \quad E[X_i] = \int_0^{\infty} t dF_{X_i}(t) = \frac{1}{\lambda_i}$$

the distribution function of the repairing time Y_i after the failure of the $i-th$ component

$$G_{X_i}(t) = P\{Y_i \leq t\} \quad E[Y_i] = \int_0^{\infty} t dF_{Y_i}(t) = \frac{1}{\mu_i}$$

Suppose X_1, X_2, Y_1, Y_2 are independent random variables.

Suppose $L(t)$denote the system state at time t, if

$$L(t) = \begin{cases} 0, \text{if two components work at time } t \\ 1, \text{if component 1 is being repaired and component 2 works at time } t \\ 2, \text{if component 2 is being repaired and component 1 works at time } t \\ 3, \text{if component 1 is being repaired and component 2 is to be repaired} \\ 4, \text{if component 2 is being repaired and component 1 is to be repaired} \end{cases}$$

then $\{L(t), t \geq 0\}$ is a stochastic process with the state space $E = \{0, 1, 2, 3, 4\}$. The working state set is $W = \{0, 1, 2\}$, and the failure state set is $F = \{3, 4\}$.From the assumption, we know $\{L(t), t \geq 0\}$ is not a MP generally. Now, introduce the supplementary variable:

$$X_i(t) = \begin{cases} \text{before time } t \text{ the last continuous working time for component } i \\ \text{up to time } t\text{, if component } i \text{ is working at time } t; \\ 0, \text{if component } i \text{ is being repaired at time } t. \end{cases}$$

$X_i(t)$ is called the lifetime for component i at time t.

$$Y_i(t) = \begin{cases} \text{the elapsed repairing time for component } i \text{ at time } t \\ \text{ if the component is being repaired} & i = 1, 2 \\ 0, \text{ if the component i is working at time t} \end{cases}$$

then $\{L(t), X_1(t), X_2(t), Y_1(t), Y_2(t)\}$ constitutes a MP and also a MSP, therefore we can determine the transient distribution of the process using the backward equation of the MSP.

Let $\tau_0 = 0$, and τ_n denotes the $n-$th time when the process $(L(t), X_1(t), X_2(t)$ $Y_1(t), Y_2(t))$ returns to state $(2, 0, 0, 0, 0)$, i.e. τ_n denotes the n-th time when component 1 just starts to work and component 2 starts the repairing process. Obviously $(L(t), X_1(t), X_2(t), Y_1(t), Y_2(t))$ is a Doob skeleton process with (τ_n) as its skeleton time sequence.

Let

$$h_j(t) = P\{L(t) = j, t \le \tau_1 | L(0) = 2, X_1(0) = X_2(0) = Y_1(0) = Y_2(0) = 0\},$$
$$M = E\{\tau_1 | L(0) = 2, X_1(0) = X_2(0) = Y_1(0) = Y_2(0) = 0\}.$$

Following the same method in proving Theorem 2.1.2, we can prove that $h_j(t)$ is uniquely determined by a minimal solution to a nonnegative linear equation system.

Theorem 4.3.1 *(1) For any* $j = 0, 1, 2, 3, 4, x_k, y_k \in [0, \infty)$, *we have*

$$P\{\tau < \infty | L(0) = j, X_1(0) = x_1, X_2(0) = x_2, Y_1(0) = y_1, Y_2(0) = y_2\} = 1.$$

(2)if and only if $\int_0^\infty tF_i(dt) < \infty, \int_0^\infty tG_i(t) < \infty, i = 1, 2,$ *for any*

$j = 0, 1, 2, 3, 4, x_k, y_k \in [0, \infty)$, *we have*

$$E\{\tau | L(0) = j, X_1(0) = x_1, X_2(0) = x_2, Y_1(0) = y_1, Y_2(0) = y_2\} < \infty.$$

thus there exists a generalized limit distribution $\Pi(\cdot)$ *for* $(L(t), X_1(t), X_2(t), Y_1(t),$

$$\Pi(j) = \frac{\int_0^\infty h_j(t)dt}{M},$$

and $\Pi(\cdot)$ *is the unique invariable probability measure for* $(L(t), X_1(t), X_2(t), Y_1(t)$

(3)if $M < \infty$ *and* $F_i(t), G_i(t)$ *is absolutely continuous, then the limit distribution of* $(L(t), X_1(t), X_2(t), Y_1(t), Y_2(t))$ *exists and equals its generalized limit distribution.*

4.4. Serial System

Suppose the system is composed of two different components and a repairman. The system is in good work state if two components work . The system is in failure state if one of the two components fails. If one component breaks down, the system is in failure state and the repairman will repair the failed components immediately and the other component will in rest state. When the failed components is repaired, the two components go back to work at once and the system enters the working state.

Suppose the two components can be restored to the original state after each failure, the distribution function of the lifetime X_i of the $i-th$

component is

$$F_i(t) = P\{X_i \leq t\},$$

the distribution function of the repairing time Y_i after the failure of the $i-th$ component is

$$G_i(t) = P\{Y_i \leq t\},$$

Suppose X_1, X_2, Y_1, Y_2 are independent random variables.

In previous study, all assume that there exist the densities of $F_i(t), G_i(t)$ $(i = 1, 2)$, and one or two of them will have negative exponential distribution. Here, we assume $F_i(t), G_i(t)$ $(i = 1, 2)$ are just general distribution.

Let $L(t)$ be the system state after time t, if

$$L(t) = \begin{cases} 0, \text{if two components work at time } t \\ 1, \text{if component1 is being repaired at time } t \\ 2, \text{if component 2 is being repaired at time } t \end{cases}$$

then $\{L(t), t \geq 0\}$ is a stochastic process with its state space $E = \{0, 1, 2\}$.The working state set of the system is $W = \{0\}$ while the failure system set is $F = \{1, 2\}$. From assumption, $\{L(t), t \geq 0\}$ is not a MP generally. Now, introduce the supplementary variables:

$$X_i(t) : \text{the lifetime for component } i \text{ at time } t,\ \ i = 1, 2$$

$$Y_i(t) = \begin{cases} \text{the elapsed repairing time for component } i \text{ at time } t, \\ \ \text{if the component is being repaired} \\ 0, \text{if the component is working at time } t \end{cases} \qquad i = 1, 2$$

then $\{L(t), X_1(t), X_2(t), Y_1(t), Y_2(t)\}$constitutes a MP and also a MSP, therefore we can determine the transient distribution of the process using the backward equation of MSP.

In general cases, there is no practical method to discuss the limit behavior of the serial system. But, when one lifetime of the components has negative exponential distribution, $\{L(t), X_1(t), X_2(t), Y_1(t), Y_2(t)\}$ becomes a Doob skeleton process, so we can deal with its limit behavior.

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Thanks!