Integration by parts formulae for the Wiener measure restricted to domains in $\mathbb{R}^d$

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- $\mathcal{C} := C([0,1];\mathbb{R}^d)$.
- For $\Omega \subset \mathbb{R}^d$ an open region,

$$D := \{w \in \mathcal{C}; w(s) \in \Omega \text{ for all } 0 \le s \le 1\}.$$

1. IbP formula for the Wiener meas. on D

$$\begin{cases} F: \text{ a smooth functional on } \mathcal{C}. \\ h = (h_1, h_2, \dots, h_d), h_i \in C_0^\infty((0,1)). \\ \text{For } a, b \in \Omega, \\ \mathcal{W}_{[0,1]}^{a,b}: \text{ the pinned Wiener meas. s.t. } w(0) = a \text{ and } w(1) = b. \\ (\cdot,\cdot): L^2([0,1])\text{-inner product.} \end{cases}$$

$$\int_D \partial_h F(w)\, d\mathcal{W}_{[0,1]}^{a,b}(w) = -\int_D F(w) \sum_{i=1}^d (h_i'', w_i)\, d\mathcal{W}_{[0,1]}^{a,b}(w) + (\mathrm{BT}),$$

where (BT) = Boundary Term

$\Rightarrow$ analogue to that in the Divergence Theorem of finite dimension.

in the connection with:

Zambotti(PTRF **123**, 2002): $\leftarrow$ { SPDEs with reflection of Nualart–Pardoux type

$$\begin{cases} \Omega = (0,\infty) \subset \mathbb{R},\ b = a \\ \text{Biane's theorem relating Brownian and 3-dim. Bessel bridges} \end{cases}$$

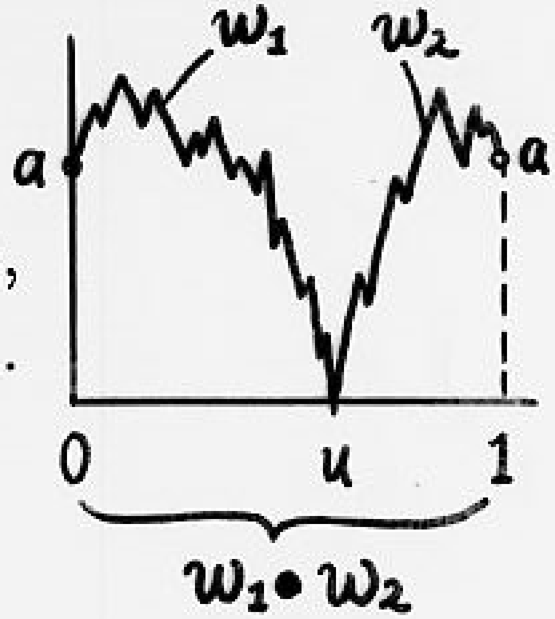
$\Rightarrow$

$$(1) \quad (\mathrm{BT}) = -\int_0^1 du\, h(u) \frac{\sqrt{2}a^2}{\sqrt{\pi u^3(1-u)^3}} \exp\left(-\frac{a^2}{2u}\right) \exp\left\{-\frac{a^2}{2(1-u)}\right\} \times E^{a,0,+}_{[0,u]} \otimes E^{0,a,+}_{[u,1]}[F(w_1 \bullet w_2)],$$

where

$E^{x,y,+}_{[l,r]}$: the expectation w.r.t. $P^{x,y,+}_{[l,r]}$, the law of 3-dim. Bessel bridge over $[l,r]$ s.t. $w(l)=x, w(r)=y$,

and



(2) $$(w_1 \bullet w_2)(s) := \begin{cases} w_1(s), & 0 \le s \le u, \\ w_2(s), & u \le s \le 1. \end{cases}$$

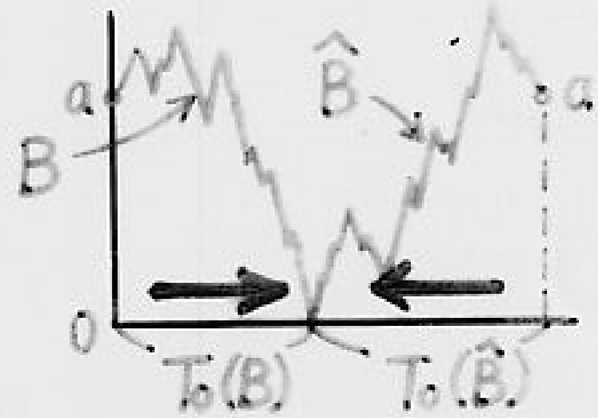
An alternative expression of (1):

- $B, \widehat{B}$: indep. Brownian motions starting at $a > 0$.
- Let

$$T_0(B) := \inf\{t > 0; B_t = 0\}, \qquad T_0(\widehat{B}) := \inf\{t > 0; \widehat{B}_t = 0\}.$$

- Conditionally on $T_0(B) + T_0(\hat{B}) = 1$, define $Y = \{Y_t, 0 \le t \le 1\}$ by

$$Y_t := \begin{cases} B_t, & 0 \le t \le T_0(B), \\ \hat{B}_{T_0(B)+T_0(\hat{B})-t}, & T_0(B) \le t \le T_0(B) + T_0(\hat{B}). \end{cases} \tag{3}$$

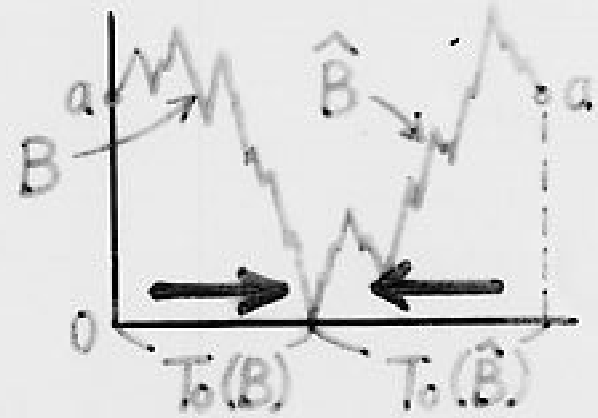


- $\mathbb{P}^{a,0,a}_{[0,1]}$: the law on $C([0,1];[0,\infty))$ of Y.
- For a path $w \in C([0,1];[0,\infty))$ starting at a and ending at b, and hitting the origin only one time, set

$$\begin{aligned} S_0(w) &:= \text{the time at which } w(S_0(w)) = 0 \\ &\overset{\text{law}}{=} T_0(B) \text{ given } T_0(B) + T_0(\hat{B}) = 1 \text{ (under } \mathbb{P}^{a,0,a}_{[0,1]}). \end{aligned}$$

- Conditionally on $T_0(B) + T_0(\hat{B}) = 1$, define $Y = \{Y_t, 0 \le t \le 1\}$ by

$$(3) \qquad Y_t := \begin{cases} B_t, & 0 \le t \le T_0(B), \\ \hat{B}_{T_0(B)+T_0(\hat{B})-t}, & T_0(B) \le t \le T_0(B) + T_0(\hat{B}). \end{cases}$$



- $\mathbb{P}^{a,0,a}_{[0,1]}$: the law on $C([0,1];[0,\infty))$ of Y.
- For a path $w \in C([0,1];[0,\infty))$ starting at a and ending at b, and hitting the origin only one time, set

$$S_0(w) := \text{the time at which } w(S_0(w)) = 0$$
$$\overset{\text{law}}{=} T_0(B) \text{ given } T_0(B) + T_0(\hat{B}) = 1 \text{ (under } \mathbb{P}^{a,0,a}_{[0,1]}).$$

With these notations,

$$(1) \doteq -4ae^{-2a^2}\mathbb{E}^{a,0,a}_{[0,1]}[h(S_0(w))F(w)]. \tag{4}$$

This follows from the fact that, conditionally on $T_0(B) = u$, $\{B_t, 0 \leq t \leq T_0(B)\}$ has the same law as $P^{a,0,+}_{[0,u]}$, and that

$$P_a(T_0(B) \in du) = \frac{a}{\sqrt{2\pi u^3}} \exp\left(-\frac{a^2}{2u}\right) du.$$

Question: What about the case of more general Ω's?

Answer: A similar expression to (4) holds.

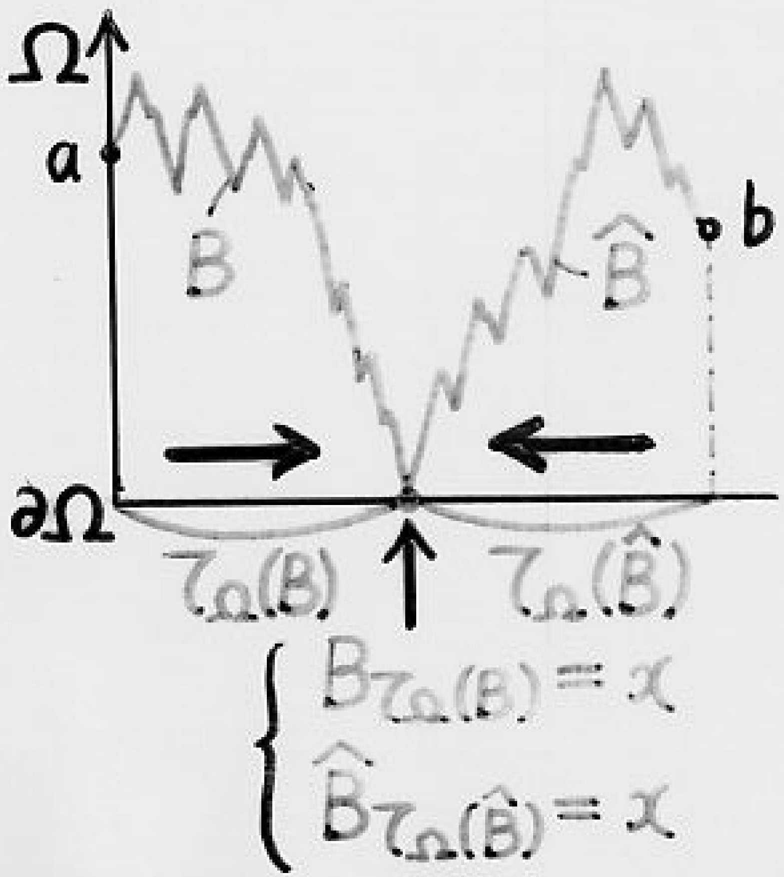
Notations:

- $\Omega \subset \mathbb{R}^d$: an open, bounded region with a piecewise smooth boundary (so that the divergence theorem in finite dimension applies to Ω).
- $B, \widehat{B}$: indep. d-dim. Brownian motions starting respectively at $a, b \in \Omega$.
- Let

$$\tau_\Omega(B) = \inf\{t > 0; B_t \in \mathbb{R}^d \backslash \Omega\}, \quad \tau_\Omega(\widehat{B}) = \inf\{t > 0; \widehat{B}_t \in \mathbb{R}^d \backslash \Omega\}.$$

- Given $\tau_\Omega(B)+\tau_\Omega(\widehat{B})=1$, $B_{\tau_\Omega(B)}=x$, and $\widehat{B}_{\tau_\Omega(\widehat{B})}=x$, $Y=\{Y_t, 0\le t\le 1\}$ is defined similarly to (3):

$$Y_t := \begin{cases} B_t, & 0 \le t \le \tau_\Omega(B), \\ \widehat{B}_{\tau_\Omega(B)+\tau_\Omega(\widehat{B})-t}, & \tau_\Omega(B) \le t \le \tau_\Omega(B)+\tau_\Omega(\widehat{B}). \end{cases}$$



- $\mathbb{P}^{a,x,b}_{[0,1]}$: the law on $C([0,1];\overline{\Omega})$ of Y.

- For a path $w \in C([0,1];\overline{\Omega})$ starting at a and ending at b, and hitting the boundary $\partial\Omega$ only one time at the point x, define

$$S_x(w) := \text{ the time at which } w(S_x(w)) = x.$$

- Δ_Ω: the Dirichlet Laplacian for Ω.
- $e^{-tH_\Omega}(y,z)$, $y,z \in \Omega$: the integral kernel of the semigroup e^{-tH_Ω} generated by $H_\Omega := -(1/2)\Delta_\Omega$; i.e.,

$$e^{-tH_\Omega}(y,z) = \frac{1}{\sqrt{(2\pi t)^d}} \exp\left\{-\frac{|y-z|^2}{2t}\right\} \mathcal{W}^{y,z}_{[0,t]}\left(w(s) \in \Omega, 0 \le s \le t\right).$$

Conditions on Ω (I still have to work on this part....):

(i) for each fixed $t > 0$ and $y \in \Omega$, the integral kernel $e^{-tH_\Omega}(y,z)$ has an extension to $\overline{\Omega}$ which is C^1 up to the boundary;

(ii) the restrictions to $\partial\Omega$ of functions which are harmonic on Ω, and C^1 up to boundary, is dense in the set of continuous functions on $\partial\Omega$.

- [Aizenman-Simon, Comm. Pure and Appl. Math. **35**]

  Theorem A. 3. 2:

$$(\mathrm{i}), (\mathrm{ii}) \Rightarrow$$

$$P_a(\tau_\Omega(B) \in dt, B_{\tau_\Omega(B)} \in dx) = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{n}_x} e^{-tH_\Omega}(a,x)\,\sigma(dx)\,dt, \tag{5}$$

where

$$\begin{cases} \sigma(dx) & : \text{ the surface measure on } \partial\Omega, \\ \boldsymbol{n}_x & : \text{ the inward normal vector at } x \in \partial\Omega, \\ \partial/\partial \boldsymbol{n}_x & : \text{ the normal derivative at } x. \end{cases}$$

Under these notations and conditions, (BT) is expressed as:

$$\text{(BT1)} \quad -\frac{(2\pi)^{d/2}}{2} e^{|a-b|^2/2} \int_{\partial\Omega} \sigma(dx)\, \mathbb{E}^{a,x,b}_{[0,1]}[\boldsymbol{n}_x \cdot h(S_x(w))F(w)] \times \int_0^1 du \frac{\partial}{\partial \boldsymbol{n}_x} e^{-uH_\Omega}(a,x) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-(1-u)H_\Omega}(b,x).$$

2. Another expression of (BT) in terms of ground state for H_Ω

- e_0: the smallest eigenvalue of $H_\Omega = -(1/2)\Delta_\Omega$; i.e.,

$$e_0 = \inf\{\frac{1}{2}\int_\Omega |\nabla f|^2\,dx;\, f \in C_0^\infty(\Omega),\, \int_\Omega |f|^2\,dx = 1\}$$
$$= -\lim_{t\to\infty}\frac{1}{t}\log P_a(\tau_\Omega(B) > t), \quad \forall a \in \Omega.$$

- f_0: a corresponding eigenfunction: $H_\Omega f_0 = e_0 f_0$.
- $\mathbb{P}_x$: the law on $C([0,\infty);\overline{\Omega})$ of X, the solution to the following SDE:

$$dX_t = dW_t + \frac{\nabla f_0}{f_0}(X_t)\,dt,\ t \geq 0, \quad X_0 = x \in \overline{\Omega}.$$

- $p_t(x,y)$: the transition density of X.
- $\mathbb{P}^{x,y}_{[l,r]}$: the pinned measure over $[l,r]$ determined from $\mathbb{P}_x$ so that

$$\mathbb{P}^{x,y}_{[l,r]}(w(l)=x, w(r)=y)=1.$$

With these notations, (BT) is also expressed as:

$$\text{(BT2)} \quad -\frac{(2\pi)^{d/2}}{2}\frac{e^{|a-b|^2/2-e_0}}{f_0(a)f_0(b)}\int_0^1 du \int_{\partial\Omega}\sigma(dx)\,h(u)\cdot \boldsymbol{n}_x\left(\frac{\partial f_0}{\partial \boldsymbol{n}_x}\right)^2 \times \mathbb{E}^{a,x}_{[0,u]}\otimes\mathbb{E}^{x,b}_{[u,1]}[F(w_1\bullet w_2)],$$

where $w_1 \bullet w_2$ is defined as (2).

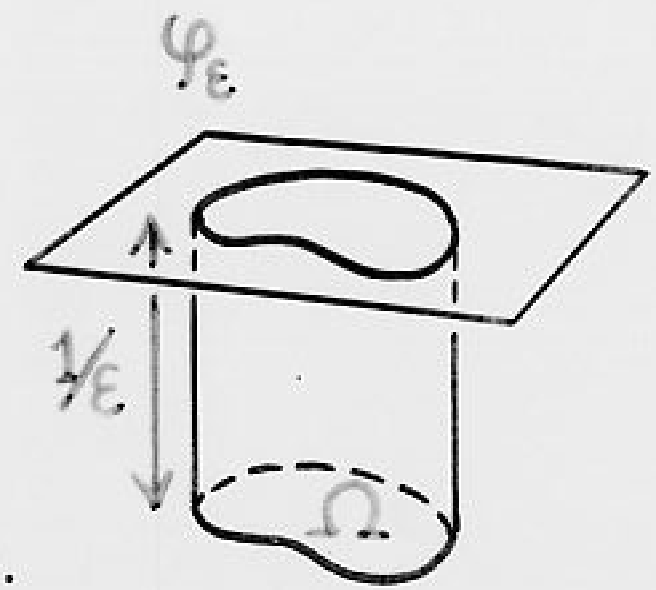
Example 1 (The case $\Omega = \{x \in \mathbb{R}; -1 < x < 1\}$).

- $f_0(x) = \cos(\frac{\pi}{2}x)$, $e_0 = \pi^2/8$.
- SDE:

$$dX_t = dW_t - \frac{\pi}{2}\tan(\frac{\pi}{2}X_t)\,dt,\ t \geq 0, \quad X_0 = x \in [-1,1].$$

$$(\mathrm{BT}) = -\left(\frac{\pi}{2}\right)^{5/2} e^{-\pi^2/8}\frac{e^{(a-b)^2/2}}{\cos(\frac{\pi}{2}a)\cos(\frac{\pi}{2}b)}$$
$$\times \int_0^1 du\, h(u)\Big\{\mathbb{E}^{a,1}_{[0,u]} \otimes \mathbb{E}^{1,b}_{[u,1]}[F(w_1 \bullet w_2)]p_u(a,1)p_{1-u}(b,1)$$
$$- \mathbb{E}^{a,-1}_{[0,u]} \otimes \mathbb{E}^{-1,b}_{[u,1]}[F(w_1 \bullet w_2)]p_u(a,-1)p_{1-u}(b,-1)\Big\}.$$

Derivation of (BT2) (heuristic):



"Penalty method" $\varphi_\varepsilon(x) := \dfrac{1}{\varepsilon} \mathbf{1}_{\Omega^c}(x)$ $(\varepsilon > 0)$.
Consider the weight:

$$G_\varepsilon(w) := \exp\{-\int_0^1 \varphi_\varepsilon(w(s))\, ds\}.$$

Cameron-Martin: $\xrightarrow[\varepsilon \downarrow 0]{} \mathbb{1}_D(w)$

$$\int_{\mathcal{C}} F(w + \delta h) G_\varepsilon(w)\, d\mathcal{W}^{a,b}_{[0,1]}(w)$$
$$= \int_{\mathcal{C}} F(w) G_\varepsilon(w - \delta h)$$
$$\times \exp\{-\delta \sum_i (h_i'', w_i) - \frac{1}{2}\delta^2 \sum_i (h_i', h_i')\}\, d\mathcal{W}^{a,b}_{[0,1]}(w).$$

Take $\left.\dfrac{d}{d\delta}\right|_{\delta=0}$ on both sides:

RHS $\to$

$$-\int_{\mathcal{C}} F(w) \sum_i (h_i'', w_i) G_\varepsilon(w)\, d\mathcal{W}^{a,b}_{[0,1]}(w)$$
$$+\sum_i \int_0^1 du\, h_i(u) \int_{\mathcal{C}} \partial_i \varphi_\varepsilon(w(u)) F(w) G_\varepsilon(w)\, d\mathcal{W}^{a,b}_{[0,1]}(w).$$

2nd term $$\text{“}=\text{”} -\int_0^1 du \int_{\partial\Omega} \sigma(dx)\, (h(u)\cdot \boldsymbol{n}_x) \frac{\mathcal{W}^{a,b}_{[0,1]}(w(u)\in dx)}{dx} \times \frac{1}{\varepsilon} \mathcal{W}^{a,x}_{[0,u]} \otimes \mathcal{W}^{x,b}_{[u,1]}[F(w_1 \bullet w_2) G_\varepsilon(w_1 \bullet w_2)].$$

(Can we confirm “$=$” by using, e.g., the theory of Watanabe's distribution?)

Rewrite:

$$\frac{1}{\varepsilon}\mathcal{W}^{a,x}_{[0,u]} \otimes \mathcal{W}^{x,b}_{[u,1]}[F(w_1 \bullet w_2)G_\varepsilon(w_1 \bullet w_2)]$$
$$= \frac{\mathcal{W}^{a,x}_{[0,u]} \otimes \mathcal{W}^{x,b}_{[u,1]}[F(w_1 \bullet w_2)G_\varepsilon(w_1 \bullet w_2)]}{\mathcal{W}^{a,x}_{[0,u]} \otimes \mathcal{W}^{x,b}_{[u,1]}[G_\varepsilon(w_1 \bullet w_2)]}$$
$$\times \frac{1}{\sqrt{\varepsilon}}\mathcal{W}^{a,x}_{[0,u]}[G_{\varepsilon,u}(w)] \times \frac{1}{\sqrt{\varepsilon}}\mathcal{W}^{x,b}_{[0,1-u]}[G_{\varepsilon,1-u}(w)]$$
$$=: I_\varepsilon \times II_\varepsilon \times III_\varepsilon,$$

where $G_{\varepsilon,t}(w) := \exp\{-\int_0^t \varphi_\varepsilon(w(s))\,ds\},\ 0 < t < 1.$

Note: $I_\varepsilon \xrightarrow[\varepsilon\downarrow 0]{} \mathbb{E}^{a,x}_{[0,u]} \otimes \mathbb{E}^{x,b}_{[u,1]}[F(w_1 \bullet w_2)].$

For II_ε and III_ε, we have:

Proposition 1. *For* $a \in \Omega$, $x \in \partial\Omega$, *and* $t > 0$,

$$\frac{1}{\sqrt{\varepsilon}} \mathcal{W}^{x,a}_{[0,t]}[G_{\varepsilon,t}(w)] \underset{\varepsilon \downarrow 0}{\longrightarrow} \frac{p_t(x,a)e^{-e_0 t}}{\sqrt{2} f_0(a)} \frac{\partial f_0}{\partial \boldsymbol{n}_x} \times (2\pi t)^{d/2} e^{|x-a|/2t}.$$

Note:

Proposition 1 $\Rightarrow$

$$II_\varepsilon \times III_\varepsilon \times \frac{\mathcal{W}^{a,b}_{[0,1]}(w(u) \in dx)}{dx}$$
$$\underset{\varepsilon \downarrow 0}{\longrightarrow} \frac{(2\pi)^{d/2}}{2} \frac{e^{|a-b|^2/2-e_0}}{f_0(a) f_0(b)} \left(\frac{\partial f_0}{\partial \boldsymbol{n}_x}\right)^2 p_u(a,x) p_{1-u}(b,x)$$

$\Rightarrow$ (BT2) follows.

Key to Proposition 1:

$$H_\varepsilon := -\frac{1}{2}\Delta + \varphi_\varepsilon \quad \text{on} \quad L^2(\mathbb{R}^d; dx).$$

ε: sufficiently small so that H_ε has an ground state f_ε.

Lemma 2.

$$\frac{1}{\sqrt{\varepsilon}} f_\varepsilon(x) \xrightarrow[\varepsilon\downarrow 0]{} \frac{1}{\sqrt{2}} \frac{\partial f_0}{\partial \boldsymbol{n}_x} \quad \textit{for } x \in \partial\Omega.$$

(f_0, f_ε: both normalized)

Remark 1. This may be regarded as a series expansion of f_ε in $\sqrt{\varepsilon}$:

$$f_\varepsilon(y) = f_0(y) + \sqrt{\varepsilon}\,(\text{1st factor}) + (\sqrt{\varepsilon})^2(\text{2nd factor}) + \cdots, \; y \in \overline{\Omega}.$$

$$\text{Lemma 2} \iff (\text{1st factor})\Big|_{\partial\Omega} = \frac{1}{\sqrt{2}} \frac{\partial f_0}{\partial \boldsymbol{n}_x}.$$

Is this kind of an expansion well-known? (WKB method?)

3. A rough sketch of the (computational) proof of (BT1)

Replace the time interval $[0,1] \to [0,t], t > 0$.
Accordingly,
$\mathcal{C} = C([0,t];\mathbb{R}^d)$,
$D = \{w \in C([0,t];\mathbb{R}^d); w(s) \in \Omega, 0 \le s \le t\}$.
In this case the IbP formula reads, for a smooth functional F on $\mathcal{C}$ and $h = (h_1, \dots, h_d), h_i \in C_0^\infty((0,t))$,

$$\text{(IbP)} \qquad \int_D \partial_h F(w)\, d\mathcal{W}^{a,b}_{[0,t]}(w)$$
$$= -\sum_{i=1}^d \int_0^t du \int_D h_i''(u) w_i(u) F(w)\, d\mathcal{W}^{a,b}_{[0,t]}(w) + (\mathrm{BT}),$$

$$
\begin{aligned}
(\mathrm{BT}) = -\frac{(2\pi t)^{d/2}}{2} \exp\left(\frac{|a-b|^2}{2t}\right) \int_{\partial\Omega} \sigma(dx)\, \mathbb{E}^{a,x,b}_{[0,t]}[\boldsymbol{n}_x \cdot h(S_x(w))F(w)] \\
\times \int_0^t du \, \frac{\partial}{\partial \boldsymbol{n}_x} e^{-uH_\Omega}(a,x) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-(t-u)H_\Omega}(b,x).
\end{aligned}
$$

Proof of (IbP):

- For a $\varphi : \mathbb{R}^d \to \mathbb{R}$ continuous, set $H^\varphi_\Omega := -(1/2)\Delta_\Omega + \varphi$.
- $e^{-tH^\varphi_\Omega}(y,z)$: the integral kernel of the semigroup $e^{-tH^\varphi_\Omega}, t > 0$:

$$
\begin{aligned}
e^{-tH^\varphi_\Omega}(y,z) = \frac{1}{\sqrt{(2\pi t)^d}} \exp\left\{-\frac{|y-z|^2}{2t}\right\} \\
\times \mathcal{W}^{y,z}_{[0,t]}[e^{-\int_0^t \varphi(w(s))\, ds}; w(s) \in \Omega, 0 \le s \le t].
\end{aligned}
$$

Lemma 3. *For $a \in \Omega$ and $x \in \partial\Omega$,*

$$E_a[e^{-\int_0^{\tau_\Omega(B)} \varphi(B_s)\,ds} | \tau_\Omega(B) = u, B_{\tau_\Omega(B)} = x] = \frac{\frac{\partial}{\partial n_x} e^{-uH_\Omega^\varphi}(a, x)}{\frac{\partial}{\partial n_x} e^{-uH_\Omega}(a, x)}.$$

Remark 2. By L'Hospital's rule, RHS is equal to

$$\lim_{\substack{z \to x \\ z \in \Omega}} \frac{\mathcal{W}_{[0,u]}^{a,z}[e^{-\int_0^u \varphi(w(s))\,ds}; w(s) \in \Omega, 0 \le s \le u]}{\mathcal{W}_{[0,u]}^{a,z}(w(s) \in \Omega, 0 \le s \le u)}.$$

Proof. Let f: a test function on $\partial\Omega$. By (5),

$$E_a[e^{-\int_0^{\tau_\Omega(B)}\varphi(B_s)\,ds} f(B_{\tau_\Omega(B)}); \tau_\Omega(B) > t]$$
$$= \frac{1}{2}\int_t^\infty du \int_{\partial\Omega} \sigma(dx)\, f(x) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-uH_\Omega}(a,x)$$
$$\times \mathbb{E}_a[e^{-\int_0^{\tau_\Omega(B)}\varphi(B_s)\,ds} | \tau_\Omega(B) = u, B_{\tau_\Omega(B)} = x].$$

So it suffices to prove

$$E_a[e^{-\int_0^{\tau_\Omega(B)}\varphi(B_s)\,ds} f(B_{\tau_\Omega(B)}); \tau_\Omega(B) > t]$$
$$= \frac{1}{2}\int_t^\infty du \int_{\partial\Omega} \sigma(dx)\, f(x) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-uH_\Omega^\varphi}(a,x). \tag{6}$$

Note: $\{\tau_\Omega(B) > t\} = \{B_s \in \Omega, 0 \le s \le t\}$. So,

LHS of (6)

$$= E_a[U(B_t)e^{-\int_0^t \varphi(B_s)\,ds}; B_s \in \Omega, 0 \le s \le t] \quad (\because \text{ Markov property})$$

$$= \int_\Omega dx\, U(x) e^{-tH_\Omega^\varphi}(a,x), \quad \text{(by definition)}$$

where $U(x) := E_x[f(B_{\tau_\Omega(B)})e^{-\int_0^{\tau_\Omega(B)} \varphi(B_s)\,ds}]$. Thus

$$\frac{\partial}{\partial t}(\text{LHS of (6)}) = \int_\Omega dx\, U(x)(-H_\Omega^\varphi)e^{-tH_\Omega^\varphi}(a,x)$$

$$= -\frac{1}{2}\int_{\partial\Omega} \sigma(dx)\, f(x) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-tH_\Omega^\varphi}(a,x)$$

$$= \frac{\partial}{\partial t}(\text{RHS of (6)}).$$

The 2nd equality follows from the divergence theorem and the fact that U solves: $H_\Omega^\varphi U = 0$ on Ω, $U|_{\partial\Omega} = f$. Q.E.D.

Back to the proof of (IbP):
Note: (IbP) is linear in F and $h = (h_i)_{1\leq i\leq d}$.

$$\Downarrow$$

Approximate F and each h_i by linear combinations of the form:

$$F(w) \sim \sum_j e^{-\int_0^t \varphi_j(w(s))\,ds}, \qquad \varphi_j: \text{ continuous},$$

$$h_i(u) \sim \sum_j k_i^j(u) l_i^j(t-u),\ 0 \leq u \leq t, \qquad k_i^j, l_i^j \in C_0^\infty((0,\infty)).$$

So it suffices to prove (IbP) for

$$F(w) = e^{-\int_0^t \varphi(w(s))\,ds} \quad \text{and} \quad h_i(u) = k_i(u) l_i(t-u), 1 \leq i \leq d:$$

$$\sum_{i=1}^{d} I_1^i(t) = \sum_{i=1}^{d} I_2^i(t) + \sum_{i=1}^{d} I_3^i(t), \tag{7}$$

where

$$I_1^i(t)$$
$$= (2\pi t)^{-\frac{d}{2}} e^{-\frac{|a-b|^2}{2t}} \int_0^t du\, k_i(u) l_i(t-u) \int_D \partial_i \varphi(w(u)) e^{-\int_0^t \varphi(w(s))\, ds}\, d\mathcal{W}_{[0,t]}^{a,b}(w),$$

$$I_2^i(t)$$
$$= (2\pi t)^{-\frac{d}{2}} e^{-\frac{|a-b|^2}{2t}} \int_0^t du \left\{ k_i''(u) l_i(t-u) + k_i(u) l_i''(t-u) - 2k_i'(u) l_i'(t-u) \right\}$$
$$\times \int_D w_i(u) e^{-\int_0^t \varphi(w(s))\, ds}\, d\mathcal{W}_{[0,t]}^{a,b}(w),$$

and by Lemma 3,

$$I_3^i(t)$$
$$= \frac{1}{2}\int_0^t du \int_{\partial\Omega} \sigma(dx)\, n_x^i k_i(u) l_i(t-u) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-uH_\Omega^\varphi}(a,x) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-(t-u)H_\Omega^\varphi}(b,x).$$

Here $\partial_i = \partial/\partial x_i$, n_x^i: the i-th component of $\boldsymbol{n}_x$.

Note:

$$I_1^i(t) = \int_0^t du\, k_i(u) l_i(t-u) \int_\Omega dx\, \partial_i \varphi(x) e^{-uH_\Omega^\varphi}(a,x) e^{-(t-u)H_\Omega^\varphi}(b,x),$$

$I_2^i(t) =$ (similar expression in terms of the integral kernel).

So, by taking the Laplace transform in t,

(7) $\iff$

$$\sum_{i=1}^{d} II_1^i(\gamma) = \sum_{i=1}^{d} II_2^i(\gamma) + \sum_{i=1}^{d} II_3^i(\gamma), \quad \gamma > 0, \tag{8}$$

where

$$II_1^i(\gamma) = \int_0^\infty\int_0^\infty ds dt\, e^{-\gamma(s+t)} k_i(s) l_i(t) \int_\Omega dx\, \partial_i\varphi(x) e^{-sH_\Omega^\varphi}(a,x) e^{-tH_\Omega^\varphi}(b,x),$$

$$II_2^i(\gamma) = \int_0^\infty\int_0^\infty ds dt\, e^{-\gamma(s+t)} \{k_i''(s) l_i(t) + k_i(s) l_i''(t) - 2k_i'(s) l_i'(t)\}$$
$$\times \int_\Omega dx\, x_i e^{-sH_\Omega^\varphi}(a,x) e^{-tH_\Omega^\varphi}(b,x),$$

$$II_3^i(\gamma) = \frac{1}{2}\int_0^\infty\int_0^\infty ds dt\, e^{-\gamma(s+t)} k_i(s) l_i(t)$$
$$\times \int_{\partial\Omega} \sigma(dx)\, n_x^i \frac{\partial}{\partial \boldsymbol{n}_x} e^{-sH_\Omega^\varphi}(a,x) \frac{\partial}{\partial \boldsymbol{n}_x} e^{-tH_\Omega^\varphi}(b,x).$$

Note: by simple integration by parts,

$$II_2^i(\gamma) =$$
$$-\int_0^\infty\int_0^\infty dsdt\, e^{-\gamma(s+t)}k_i(s)l_i(t)\int_\Omega dx\left\{(H_\Omega^\varphi e^{-sH_\Omega^\varphi}(a,x))(\partial_i e^{-tH_\Omega^\varphi}(b,x)) + (\partial_i e^{-sH_\Omega^\varphi}(a,x))(H_\Omega^\varphi e^{-tH_\Omega^\varphi}(b,x))\right\}.$$

Compare these expressions ⇒ (8) is reduced to:

Proposition 4. *For functions f, g on $\overline{\Omega}$ s.t. $f|_{\partial\Omega} = g|_{\partial\Omega} = 0$, and for $v \in \mathbb{R}^d$,*

$$\int_\Omega dx\,(v\cdot\nabla\varphi)fg = -\int_\Omega dx\,\{(v\cdot\nabla g)H_\Omega^\varphi f + (v\cdot\nabla f)H_\Omega^\varphi g\} + \frac{1}{2}\int_{\partial\Omega}\sigma(dx)\,(v\cdot \boldsymbol{n}_x)\frac{\partial f}{\partial \boldsymbol{n}_x}\frac{\partial g}{\partial \boldsymbol{n}_x}. \tag{9}$$

Proof. We only need to consider the case $g = f$. Then by the definition of H^{φ}_{Ω},

$$\text{1st term on RHS} = \int_{\Omega} dx\,(v \cdot \nabla f)\Delta f - 2\int_{\Omega} dx\,(v \cdot \nabla f)\varphi f.$$

Note: 2nd term = LHS of (9); indeed,

$$\begin{aligned}
&\int_{\Omega} dx\,(v \cdot \nabla \varphi)|f|^2 + 2\int_{\Omega} dx\,(v \cdot \nabla f)\varphi f \\
&\quad = \int_{\Omega} dx\,\mathrm{div}(\varphi |f|^2 v) \\
&\quad = -\int_{\partial\Omega} \sigma(dx)\,(v \cdot \boldsymbol{n}_x)\varphi |f|^2 && (\because\ \text{divergence theorem}) \\
&\quad = 0. && (\because\ f|_{\partial\Omega} = 0)
\end{aligned}$$

So It remains to prove:

$$\int_\Omega dx\,(v\cdot\nabla f)\Delta f + \frac{1}{2}\int_{\partial\Omega}\sigma(dx)\,(v\cdot\boldsymbol{n}_x)\left|\frac{\partial f}{\partial \boldsymbol{n}_x}\right|^2 = 0. \tag{10}$$

Apply the divergence theorem to $(v\cdot\nabla f)\nabla f$ to get:

(11)

$$\begin{aligned}
&\int_\Omega dx\,(v\cdot\nabla f)\Delta f + \sum_{i,j=1}^{d}\int_\Omega v_i(\partial_i\partial_j f)\partial_j f \\
&\quad = -\int_{\partial\Omega}\sigma(dx)\,(v\cdot\nabla f)\frac{\partial f}{\partial \boldsymbol{n}_x} \\
&\quad = -\int_{\partial\Omega}\sigma(dx)\,(v\cdot\boldsymbol{n}_x)\left|\frac{\partial f}{\partial \boldsymbol{n}_x}\right|^2. \qquad \left(\because\ \nabla f = \frac{\partial f}{\partial \boldsymbol{n}_x}\boldsymbol{n}_x \text{ at } x\in\partial\Omega\right)
\end{aligned}$$

Moreover,

$$\text{(12)} \qquad \sum_{i,j=1}^{d} \int_\Omega dx\, v_i(\partial_i\partial_j f)\partial_j f = \frac{1}{2}\int_\Omega dx\, \mathrm{div}(|\nabla f|^2 v)$$
$$= -\frac{1}{2}\int_{\partial\Omega} \sigma(dx)\,(v\cdot \boldsymbol{n}_x)|\nabla f|^2$$
$$= -\frac{1}{2}\int_{\partial\Omega} \sigma(dx)\,(v\cdot \boldsymbol{n}_x)\left|\frac{\partial f}{\partial \boldsymbol{n}_x}\right|^2 .$$

Combining (11) and (12) yields (10) and ends the proof of the proposition. Q.E.D.