SCALING LIMITS OF EQUILIBRIUM WETTING MODELS IN (1+1) DIMENSION

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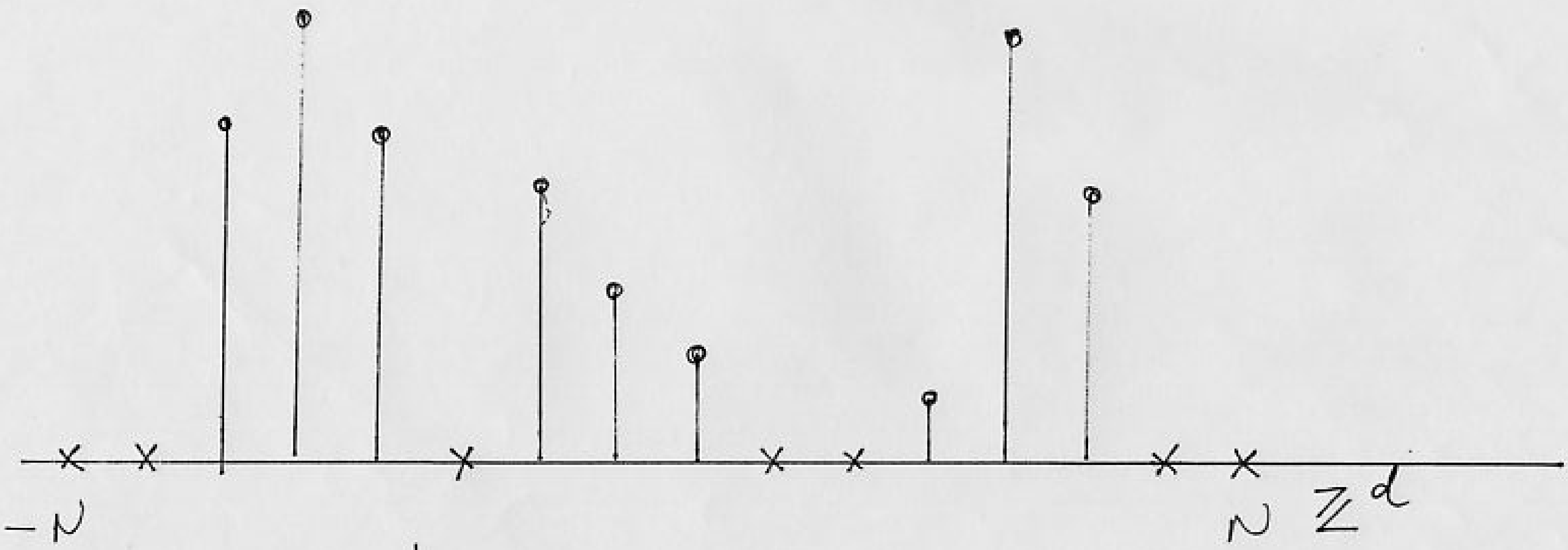
1. WETTING TRANSITION

INTERFACE MODEL



$i : \mathbb{Z}^d \longrightarrow X_i \in \mathbb{R}_+$

HEIGHT OF INTERFACE
ABOVE i

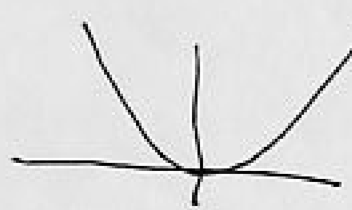
GIBBS MEASURE ON $V_N = [-N, N]^d$

$$P^+_{\varepsilon, N}(dx) = \frac{\exp(-H_N(x))}{Z^+_{\varepsilon, N}} \prod_{i \in V_N} (dx_i^+ + \varepsilon \delta_0) \prod_{i \in V_N^c} \delta_0$$

$$H_N(x) = \sum_{\langle i,j \rangle} V(x_i - x_j) \,,$$ HAMILTONIAN

$V : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ POTENTIAL

$\varepsilon > 0$ PINNING PARAMETER



$Z^+_{\varepsilon, N}$ CONSTANT

REWRITE

$$P^+_{\varepsilon,N}(dx) = \sum_{A \subseteq V_N} p^+_{\varepsilon,N}(A) \cdot P^+_{0,A^c}(dx)$$

WHERE

$$P^+_{0,A^c}(dx) = \frac{1}{Z^+_{0,A^c}} \exp(-H_{A^c}(x)) \prod_{i \in A^c} dx_i^+ \prod_{i \in A} \delta_0$$

Distribution without pinning

$$p^+_{\varepsilon,A}(A) = \frac{\varepsilon^{|A|} \cdot Z^+_{0,A^c}}{Z^+_{\varepsilon,N}}$$

Distribution of pinned sites

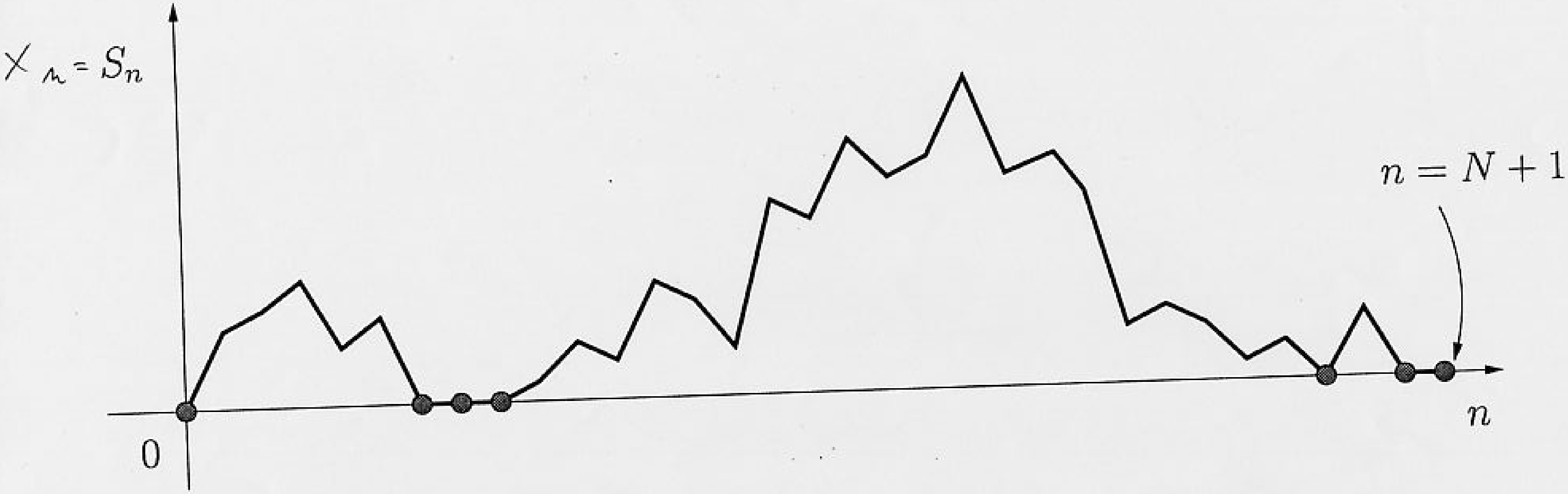


FIGURE 1. A trajectory of $\mathbf{P}^{c}_{\varepsilon,31}$ with 5 pinned sites (0 and $N+1 = 32$ are not counted). We interpret the thick line joining the points (n, S_n) as the interface separating a liquid phase (below) from a gas phase (above). The interface is constrained to stay above a flat wall, to which is however attracted by a potential that acts only *very close* to the wall. Pinned sites are therefore called also *dry* sites, while the complementary set is identified as *wet* set or region.

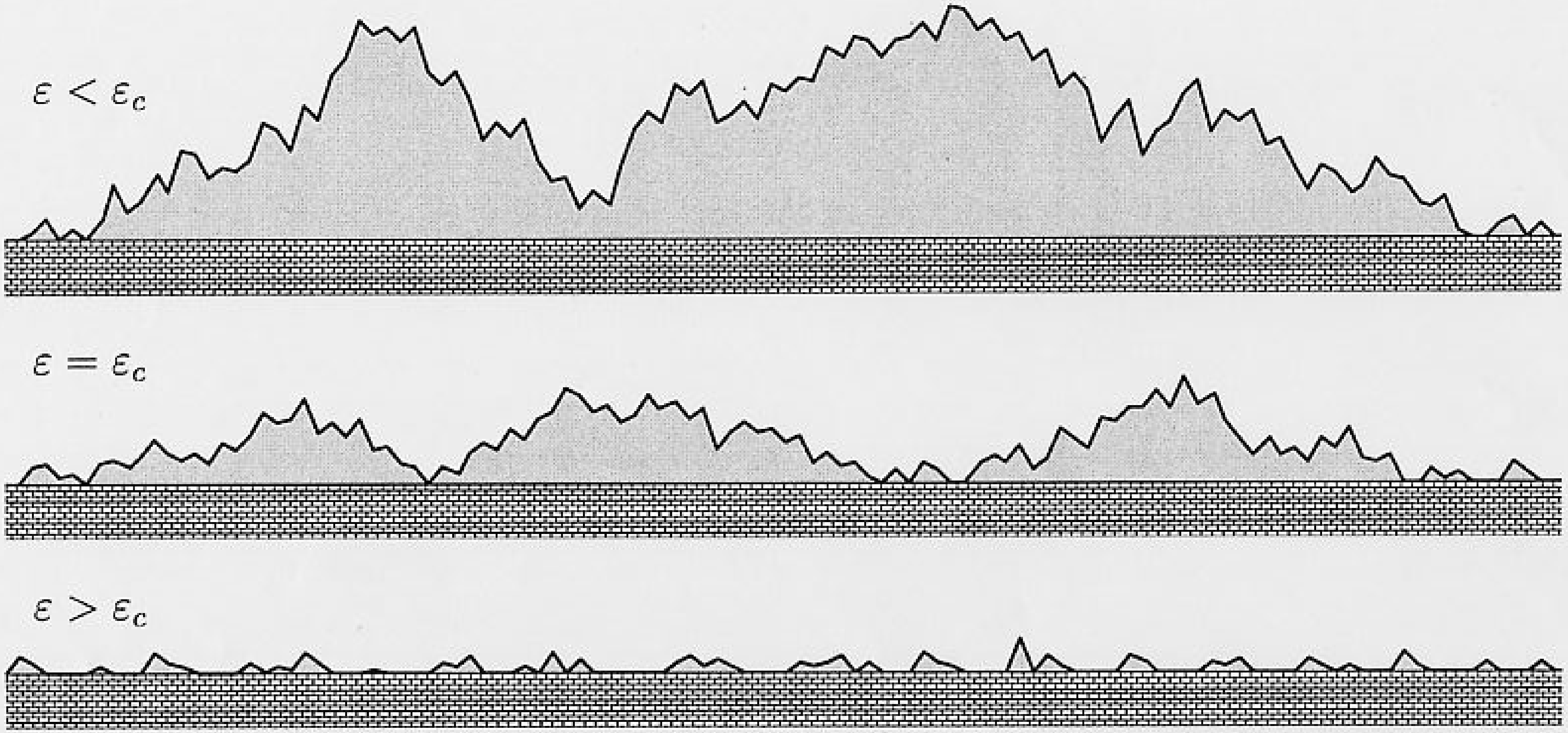


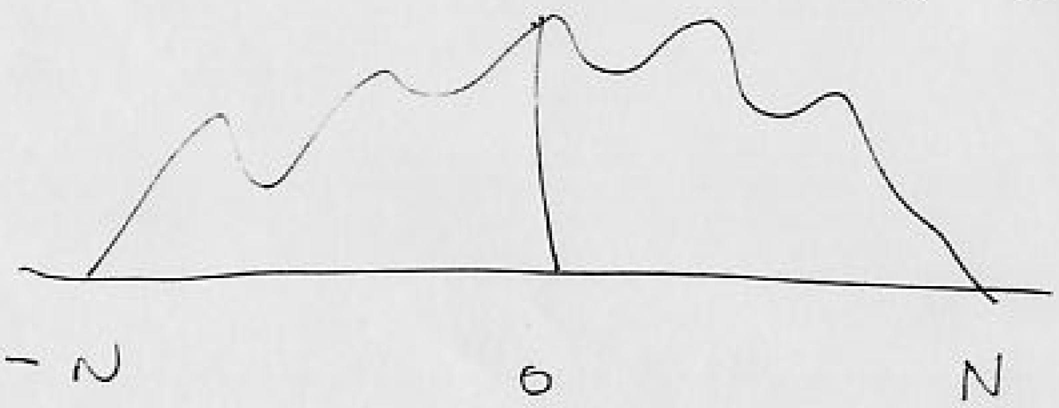
FIGURE 2. A visual representation for the constrained case of the three regimes identified in Theorem 1.1 and sharpened in Theorem 1.2 and Proposition 1.3. In the subcritical regime the dry set is confined to a neighborhood of the boundary, while in the supercritical regime the dry region is spread through the bulk of the system and the wet regions are organized in small droplets. In the critical case the diffusive scaling is non–trivial, like in the subcritical regime, and the wet region is *mostly* made up by droplets of size comparable to the size of the whole system: the dry set has zero density, but it is spread through the bulk of the system.

ENTROPIC REPULSION

$\varepsilon = 0$ NO PINNING

$$\mathbb{E}^{+}_{0,N}[X_0] \approx \begin{cases} N^{1/2} & d=1 \\ \log N & d=2 \\ \sqrt{\log N} & d\geq 3 \end{cases}$$

$$\mathrm{var}^{+}_{0,N}(X_0) \approx \begin{cases} N & d=1 \\ \log N & d=2 \\ O(1) & d\geq 3 \end{cases}$$



WETTING TRANSITION

FIND $\varepsilon_0 > 0$ SUCH THAT

$\varepsilon > \varepsilon_0$ STRONG PINNING $\longrightarrow$ LOCAL.

$\varepsilon \leq \varepsilon_0$ WEAK PINNING $\longrightarrow$ DELOCAL.

THEOREM $d=1$ FISHER, $d=2$ CAPUTO - VELENIK
$d \geq 3$ BOLTHAUSEN, ZEITOUNI, Δ

LET $c_1 \leq V'' \leq c_2$, $d \geq 3$ $\Rightarrow$ $\varepsilon_0 = 0$ NO WETT.

$d = 1, 2$ $\varepsilon_0 > 0$ WETTING!

2. 1+1 MODEL

NOTATION :

$$P^+_{\varepsilon,N}(dx) = \frac{1}{Z^+_{\varepsilon,N}} \exp(-H_N(x)) \prod_{i=1}^{N} \left(dx_i^+ + \varepsilon\, \delta_0\right)$$

$$H_N(x) = \sum_{i=0}^{N} V(x_{i+1} - x_i) \quad , \quad x_0 = x_{N+1} = 0$$

$\{Y_n\}_{n\in\mathbb{N}}$ iid with density $e^{-V(y)}\,dy$

i.e. $\int_{\mathbb{R}} e^{-V(y)}\,dy = 1$

Set $X_n = Y_1 + Y_2 \cdots + Y_n$ RW on $\mathbb{R}$

Set $\Omega_N^+ = \{ x_n \geq 0 , \; n = 0, 1, \ldots, N \}$

THEN

$$\mathbb{P}^+_{0,N}(\,\cdot\,) \equiv \mathbb{P}(\;\cdot\; \mid \Omega_N^+ \cap \{X_0 = 0, X_{N+1} = 0\})$$

$$= \frac{\exp\left(-\sum_{n=0}^{N} V(X_{n+1} - X_n)\right)}{Z^+_{0,N}} \prod_{n=1}^{N} dX_n^+$$

<u>ASSUME</u> $\exp(-V) \in C(\mathbb{R} \cup \{+\infty\})$

$$\mathbb{E}[\,Y_n\,] = 0 \quad , \quad \mathbb{E}[\,|Y_n|^3\,] < \infty$$

SET $\quad \sigma^2 \equiv \mathrm{Var}(Y_n)$

THEOREM Let

$$X_t^{(N)} = \frac{X_{[Nt]}}{\sigma N^{1/2}} + (Nt - [Nt]) \frac{X_{[Nt]+1} - X_{[Nt]}}{\sigma N^{1/2}}, \quad t \in [0,1]$$

Set

$$\varepsilon_0 = \frac{1}{1 + \sum_{n=1}^{\infty} z^+_{0,n}}$$

1) $\forall \; \varepsilon < \varepsilon_0$

$$P^+_{\varepsilon,N} \circ X^{(N)\,-1} \Longrightarrow$$ LAW OF BROWNIAN EXCURSION

2) $\varepsilon = \varepsilon_0$

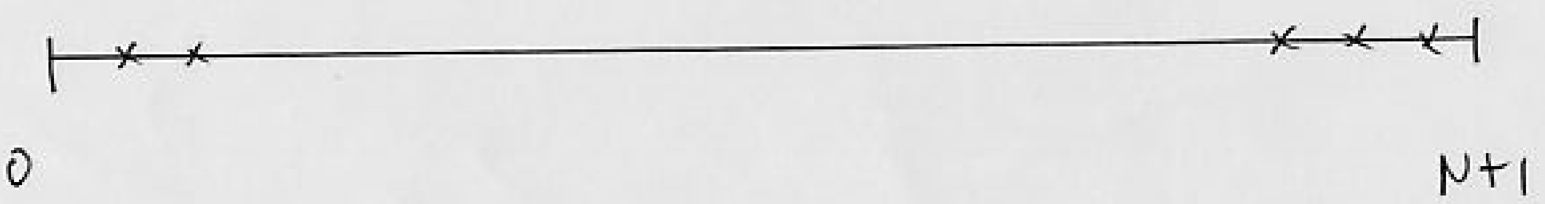
$$P^+_{\varepsilon,N} \circ X^{(N)\,-1} \Longrightarrow$$ LAW OF REFLECTED BROWNIAN BRIDGE

DRY SITES

$$A = \{ i : X_i = 0 \} = \{ 0 = t_0 < t_1 \dots < t_{|A|} < t_{|A|+1} = N+1 \}$$

THEOREM $\forall \, \varepsilon < \varepsilon_0$

$$\lim_{L \to \infty} \limsup_{N \to \infty} P^{+}_{\varepsilon, N} \left(A \cap [L, N-L] \neq \emptyset \right) = 0$$



$$P^{+}_{\varepsilon,N}(dx) = \sum_{A \subseteq V_N} P^{+}_{\varepsilon,N}(A) \bigotimes_{j=1}^{|A|+1} P^{+}_{0,[t_{j-1},t_j]}(dx)$$

$$P^{+}_{\varepsilon,N}(A) = \frac{\varepsilon^{|A|}}{Z^{+}_{\varepsilon,N}} \prod_{i=1}^{|A|+1} Z_{t_i - t_{i-1}}$$

$$Z_1 = 1 \qquad Z_n = Z^{+}_{0,n-1} = \int_{\mathbb{R}_+^{n-1}} e^{-H_{n-1}(x)} \prod_{i=1}^{n-1} dx_i^{+}$$

Set $\gamma = \sum_{n=1}^{\infty} Z_n < \infty$, $\quad q(n) = \frac{Z_n}{\gamma}$, $\quad \delta = \varepsilon\gamma$

$$\boxed{P^{+}_{\delta,N}(A) = \frac{\delta^{|A|+1}}{\tilde{Z}_{\delta,N+1}} \prod_{i=1}^{|A|+1} q(t_i - t_{i-1})}$$

<u>LEMMA</u> $\quad Z_n = \frac{1}{\sqrt{2\pi\sigma^2}} n^{-3/2} + o(n^{-3/2})$

THUS $\quad q(n) = c\, n^{-3/2} + o(n^{-3/2})$

<u>PROOF</u>

$$Z_{n+1} = \int_{\mathbb{R}_+^n} \exp(-H_n(x)) \prod_{i=1}^{n} dx_i^+$$

$$= P_{0,n}(\Omega_n^+) \cdot \mathbb{E}\Big[\exp\Big(-V\Big(\sum_{i=1}^{n} Y_i\Big)\Big)\Big]$$

$$P_{0,n}(\Omega_n^+) = \frac{1}{n+1}$$ exchangeable !

$$\mathbb{E}[\exp(-V(X_n))] = \frac{1}{\sqrt{2\pi\sigma^2}} n^{-1/2} + o(n^{-1/2})$$

LOCAL CLT !

LEMMA

$$\tilde{Z}^{+}_{\delta,N} = \delta \sum_{t=1}^{N} q(t)\, \tilde{Z}^{+}_{\delta,N-t}$$

is GREEN FUNCTION OF RW ON $\mathbb{N}_+$ with JUMPS $\left(q(n)\right)_{n\in\mathbb{N}_+}$ AND KILLING δ

$$\Longrightarrow \quad \delta < 1 \qquad \tilde{Z}^{+}_{\delta,N} \approx n^{-3/2}$$

$$\delta = 1,\ \varepsilon = \varepsilon_0 \qquad \tilde{Z}^{+}_{\delta,n} \approx n^{-1/2}$$

COROLLARY

$\varepsilon < \varepsilon_0$ (SUBCRITICAL CASE)

$$\lim_{L\to\infty} \lim_{N\to\infty} P^{+}_{\varepsilon,N}\left(A \cap [L, N-L] \neq \phi \right) = 1$$

PROP Let e = BROWNIAN EXCURSION

$$P^{+}_{0,N} \circ X^{(N)\,-1} \Longrightarrow \text{LAW OF } e$$

PROOF INVARIANCE PRINCIPLE

LET $\tau \in [0,1]$ $\quad B(\tau) = \min\limits_{s \in [0,1]} B(s)$

$$e(t) = B(\tau \oplus t) - B(\tau) \qquad 0 \le t \le 1$$

$$\mathcal{L}\Big(\big(X_{n \oplus T} - X_T \big)_{n=0,\dots,N+1} \Big) = P^{+}_{0,N}$$

$T \in \{0, \dots N\}$ $\quad X_T = \min X_n$

SINCE $(Y_n)_{n=0,\dots,N}$ exchangeable !

THEOREM $\varepsilon = \varepsilon_0$ CRITICAL CASE

Let $A^{(N)} = A/_N$

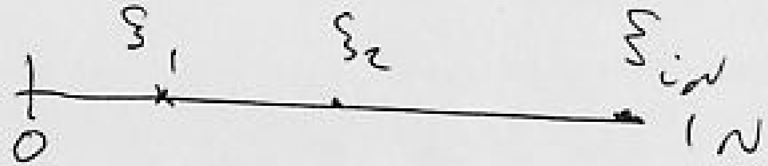
$P^{+}_{\varepsilon_0, N} \circ A^{(N)-1} \Longrightarrow$ LAW of $\{t \in [0,1], \beta(t) = 0\}$

β is BROWNIAN BRIDGE

PROOF USE REGENERATIVE SETS !

Let $(\xi_n)_{n \in \mathbb{N}}$ RW on $\mathbb{N}_+$ with $(q(n))_{n \in \mathbb{N}}$

$i_N = \text{Max}\{ k : \xi_k \le N \}$



$\tilde{A}^{(N)} = \tilde{A}/_N$ WHERE $\tilde{A} = \{ \xi_1, . \xi_{i_N} \}$

NOTE $\tilde{A} = \{ t_1, \dots, t_{|A|} \}$

$$P(\tilde{A}) = \prod_{j=1}^{|A|+1} q(t_j - t_{j-1}) \, Q(N - t_{|A|}), \quad Q(k) = \sum_{n > k} q(n)$$

SINCE $q(n) = c\, n^{-3/2} + o(n^{-3/2})$

LAW of $\tilde{A}^{(N)} \Longrightarrow$ LAW of $\{t \in [0,1];\ B(t) = 0\}$

$(B(t))$ BROWNIAN MOTION

FOR $A^{(N)}$: CONDITIONED RETURN TO 0

AT TIME N+1 !

ONCE **DRY** SITES KNOWN

USE EXCURSION THEORY!

3. DYNAMIC

SDE WITH STICKY DIFFUSION:

$$dX_i(t) = -\sum_{j:|i-j|=1} V'\big(X_i(t) - X_j(t)\big)\,\mathbb{1}_{X_i(t)>0}\,dt$$

$$+\ d\ell_i(t)\ +\ \sqrt{2}\,\mathbb{1}_{X_i(t)>0}\,dw_i(t)\ ,\ i \in V_N$$

a) $X_i(t) \geq 0$, $d\ell_i(t) \geq 0$, $\ell_i(0) = 0$

b) $\int_0^\infty X_i(s)\,d\ell_i(s) = 0$

c) $\varepsilon\ \ell_i(t) \underset{\uparrow}{=} \int_0^t \mathbb{1}_{\{0\}}(X_i(s))\,ds$

STICKY CONDITION

MARTINGALE PROBLEM:

$F \in C_b^2(\mathbb{R}_+^{V_N})$

$$F(X_t) - \int_0^t \mathcal{L} F(X_s)\, ds - \sum_{i \in V_N} \varepsilon \int_0^t L F(X_{s,i})\, d\ell_i(s)$$

is MARTINGALE

WHERE

$$\mathcal{L} F(x) = \sum_{i \in V_N} \left\{ -\partial_{x_i} H_N(x)\, \partial_{x_i} F(x) + \partial_{x_i}^2 F(x) \right\} = \sum_i \mathcal{L}_i F(x)$$

$$LF(x,i) = \frac{1}{\varepsilon} \partial_{x_i} F(x) - \mathcal{L}_i F(x) \quad , \quad x_i = 0$$

BOUNDARY OPERATOR

PROP (FUNAKI - D) $\quad P^+_{\varepsilon, N}$ is REVERSIBLE GIBBS

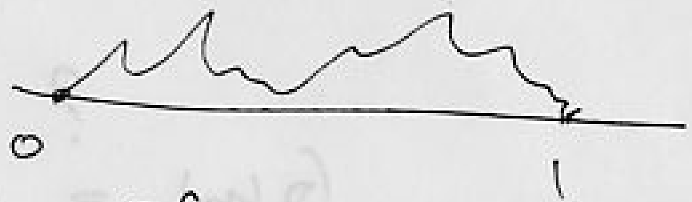
FOR DYNAMIC

TAKE FIRST $\varepsilon = 0$, NO PINNING
JUST REPULSION

LAW of $X(0) = P^{+}_{0,N}$ EQUILIBRIUM

SET $X^{(N)}(t,\theta) = N^{-1/2} X_{[\theta N]}(tN^2)$

$0 \le t \le 1$, $0 \le \theta \le 1$



PROP. FUNAKI - OLLA, ZAMBOTTI, If $\varepsilon < \varepsilon_0$ Z&D

LAW of $X^{(N)} \Longrightarrow$ LAW of $(Z(t,\theta))$

SOLUTION OF SPDE WITH REFLECTION OF NUALARD PARDOUX:

$$\frac{\partial}{\partial t} Z(t,\theta) = c \frac{\partial^2}{\partial \theta^2} Z(t,\theta) + \sqrt{2}\, \dot{W}(t,\theta) + \xi(t,\theta)$$

$Z(t,\theta) \geq 0$, $\iint Z(t,\theta)\, \xi(dt,d\theta) = 0$, $Z(t,0) = Z(t,1) = 0$

RW on $\mathbb{N}$

$$q(n) = c\, n^{-3/2} + o(n^{-3/2})$$

$$G(n) = \sum_{t=1}^{n} q(t)\, G(n-t)$$

$$G(n) \overset{?}{=} c'\, n^{-1/2} + o(n^{-1/2})$$

以上是报告的全部内容。

謝谢大家！