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The Reversible Nearest Particle System on a Finite Set

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Nearest Particle System (NPS)

$S \subset \mathbb{Z}$, $\quad X = \{0,1\}^S$

$$\xi \in X \rightleftharpoons A \subset S, \qquad A = \{x : \xi(x) = 1\}$$

transition rate

$x \in A$, $A \to A \setminus \{x\}$ with rate 1

$x \in S \setminus A$, $A \to A \cup \{x\}$ with rate $\beta(l_x(A), r_x(A))$

ξ_t^N, NPS on $\{0, 1, \cdots, N\}$

hitting time

$$\sigma_N := \inf\{t \geq 0 : \xi_t^N = \emptyset\}$$

What is the rate that $\sigma_N \uparrow \infty$?

Contact process

$$\beta(l,r) = \begin{cases} 2\lambda, & l = r = 1; \\ \lambda, & l = 1, r > 1 \text{ or } l > 1, r = 1; \\ 0, & \text{otherwise} \end{cases}$$

η_t^N, contact process on $\{0, 1, \cdots, N\}$

hitting time, $\tau_N := \inf\{t \geq 0 : \eta_t^N = \emptyset\}$

Critical value on $\mathbb{Z}$

$$\lambda_c' := \inf\{\lambda : P^\eta(\eta_t \neq \emptyset \text{ for all } t \geq 0) > 0\}$$

Theorems about contact process

Durrett & Liu, *Ann. Probab.* (1988) If $\lambda > \lambda_c'$, then $\exists\, \gamma_1 > 0$ s.t.

$$\frac{\tau_N}{\log N} \to \frac{1}{\gamma_1} \quad \text{in prob.}, \quad N \to \infty$$

Durrett & Schonmann, *Ann. Probab.* (1988) If $\lambda' > \lambda_c'$, then $\exists\, \gamma_2 > 0$ s.t.

$$\frac{\log \tau_N}{N} \to \gamma_2 \quad \text{in prob.}, \quad N \to \infty$$

Durrett, Schonmann & Tanaka, *Ann. Probab.* (1989) If $\lambda' = \lambda_c'$, $a, b \in (0, \infty)$, then

$$P\left(aN \le \tau_N \le bN^4\right) \to 1, \quad N \to \infty$$

reversible measure for ξ_t^N

$$\pi(A)q(A,B) = \pi(B)q(B,A), \text{ for all } A, B \neq \emptyset.$$

Suppose $\beta(\cdot) > 0$, $\lambda = \sum_{l=1}^{\infty} \beta(l) < \infty$, and

$$\beta(l,r) = \frac{\beta(l)\beta(r)}{\beta(l+r)}, \quad \beta(l,\infty) = \beta(\infty,l) = \beta(l).$$

Then

$$\pi(A) = 1, \quad A = \{x\};$$
$$\pi(A) = \prod_{i=1}^{n-1} \beta(x_{i+1} - x_i), \quad A = \{x_1, x_2, \cdots, x_n\}$$

$\boldsymbol{\xi_t}$ survives if $\boldsymbol{\lambda = \sum_{l=1}^{\infty} \beta(l) > 1}$, and dies otherwise.

$$\boldsymbol{P^{\xi}(\xi_t \neq \emptyset \text{ for all } t \geq 0) > 0 \Leftrightarrow \lambda > 1}$$

T.M.Liggett, *Interacting Particle Systems*, Springer, 1985.

Consider one-parameter family

$\boldsymbol{\beta_\lambda(l) = \lambda\psi(l)}$, where $\boldsymbol{\psi(l) > 0}$, $\boldsymbol{\sum_{n=1}^{\infty} \psi(n) = 1}$.

Attractive: $\boldsymbol{\frac{\psi(n)}{\psi(n+1)} \downarrow 1}$.

Ideas for estimating σ_N from above —

coupling with a birth and death process

$$\xi_t^N = \emptyset \Leftrightarrow |\xi_t^N| = 0$$

$$|\xi_t^N| = n \quad \to n-1 \quad \text{with rate } n$$

$$\to n+1 \quad \text{with rate } \le (n+1)M$$

$n+1$: the number of intervals

M : the supreme rate of birth in an interval

Theorem 1

Suppose

$$M \stackrel{\triangle}{=} \sup_n \sum_{l+r=n} \frac{\psi(l)\psi(r)}{\psi(n)} < \infty.$$

Then for any $C_N \to \infty$,

$$\lim_{N\to\infty} P\big(\sigma_N \le C_N f_\lambda(N)\big) = 1,$$

where

$$f_\lambda(N) = \begin{cases} \log N & \text{if } \lambda M < 1; \\ N \log N & \text{if } \lambda M = 1; \\ (\lambda M)^N & \text{if } \lambda M > 1. \end{cases}$$

Estimating $\boldsymbol{\sigma_N}$ from below — coupling with $\boldsymbol{\beta(l, r) = 0}$

$$\lim_{N\to\infty} P\big(\boldsymbol{\sigma_N} > (1-\varepsilon)\log N\big) = 1, \quad \forall\, \varepsilon > 0.$$

$\boldsymbol{\lambda} \ll 1, \qquad \boldsymbol{\sigma_N} \sim \log N$

$\boldsymbol{\lambda} \gg 1, \qquad \boldsymbol{\sigma_N} \sim e^{\gamma N}$?

Ideas for estimating σ_N from below —

coupling with a contact process, renormalization

$$\{0,1,\cdots,N\} \sim I_1 \cup I_2 \cup \cdots \cup I_n, \quad |I_i| = N_0.$$

Construct a contact process η_t^k (with infection rate λ') such that

$$\sum_{x \in I_k} \xi_t^N(x) \geq \eta_t^n(k).$$

$$\eta_0^n(k) = \begin{cases} 1 & \text{if } \sum_{x\in I_k} \xi_0^N(x) \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

coupling ξ_t^N on I_k with $\eta_t^n(k)$ as:

1. If $\sum_{x \in I_k} \xi_t^N(x) = \eta_t^n(k) = 1$, then

$$\xi_t^N(x) \to 0 \Rightarrow \eta_t^n(k) \to 0.$$

2. If $\sum_{x \in I_k} \xi_t^N(x) = \eta^n(k) = 0$, then

$$\eta_t^n(k) \to 1 \Rightarrow \xi_t^N(x) \to 0 \quad \text{for some } x \in I_k.$$

$$\lambda' \leq \textstyle\sum_{x \in I_k} \text{birth rate at } x.$$

Theorem 2.

Suppose

$$\lambda > \lambda_c' \max\left\{ \frac{2\psi(3n_0)}{\sum_{l=n_0}^{2n_0}\psi(l)\psi(3n_0-l)},\ \frac{1}{\sum_{l=n_0}^{2n_0}\psi(l)} \right\}$$

for some n_0. Then $\exists\, \gamma > 0$ s.t.

$$\lim_{N\to\infty} P\left(\sigma_N \geq e^{\gamma N}\right) = 1.$$

Critical case : $\lambda = 1$

Ideas for estimating σ_N from above — coupling with a NPS on $\mathbb{Z}$

ξ_t : critical NPS on $\mathbb{Z}$.

ξ_0 : $\xi_0(0) = 1$, $\xi_0|_{\mathbb{Z}^-} = 0$,

$$P(\xi_0|_{\{0,1,\cdots,N\}}) \sim \mathrm{Ren}(\beta) = \pi(\cdot|\xi(0) = 1)$$

$$r_{a^2t}/a \to B_t, \quad a \to \infty$$

Schinazi, *Ann. Probab.* (1992)

$$\xi_t \geq \xi_t^n, \qquad r_t < -n \Rightarrow \xi_t^n = \emptyset, \quad \forall\, n$$

Ideas for estimating σ_N from below —

modified to be reversible at $\emptyset$

$\tilde{\xi}_t^N$: NPS on $\{0, 1, \cdots, N\}$ modified as:

$$\emptyset \longrightarrow \{x\} \text{ with rate } q > 0$$

To be a reversible process: $\tilde{\pi}(\{\emptyset\}) = q^{-1}, \tilde{\pi}(A) = \pi(A)$

$$\tau = \inf\{t \geq 0 : \tilde{\xi}_t^N = \emptyset\}$$

$$K_N = \sum_{A \in \mathcal{S}_N \setminus \{\emptyset\}} \pi(A) \approx N^2$$

Theorem 3

Suppose $\boldsymbol{\lambda = 1}$, $\boldsymbol{\xi_0^N \sim \mathrm{Ren}(\beta)}$.

Then for $\boldsymbol{C_N \to 0}$ and $\boldsymbol{C'_N \to \infty}$,

$$\lim_{N \to +\infty} P\left(C_N N \leq \sigma_N \leq C'_N N^2\right) = 1.$$

Thank you!

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