Filtration Consistent Nonlinear Expectations and Evaluations

August 14, 2004

1. Filtration Consistent Nonlinear Expectations and Evaluations

PENG Shige
Institute of Mathematics
Institute of Finance
Shandong University

The 3rd Workshop on
MARKOV PROCESSES AND RELATED TOPICS
Beijing, August $10-14, 2004$

- Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration.
  $\mathcal{F}_t$ represents the information acquired by an economic agent,
  i.e.,(an individual, a firm, or a market) during the period $[0,t]$.
- $X \in m\mathcal{F}_t :=$ the collection of $\mathcal{F}_t$–measurable random variables.
  Example: $X = \max\{S_t - q, 0\}$: an option with the maturity t.
- At the present time s, this agent evaluates a future risky payoff X (e.g. an option) with maturity $t \geq s$.
- His payoff at time t is X. This money based value that will be known at the time t: $X \in m\mathcal{F}_t$.

Problem: At time s, How much he will pay to buy this X?

- We denote his evaluation of X at time s by $\mathcal{E}_{s,t}[X]$ ($\in m\mathcal{F}_s$)
- We then have a family of mappings

$$\mathcal{E}_{s,t}[\cdot] : m\mathcal{F}_t \to m\mathcal{F}_s,\ 0 \le s \le t < \infty$$

We make the following:

1.1. Axiomatic assumptions:

for each $0 \le r \le s \le t$, for each X, $X' \in m\mathcal{F}_t$,

(**A1**) $\mathcal{E}_{s,t}[X] \ge \mathcal{E}_{s,t}[X'],\ if\ \ X \ge X'$;
(**A2**) $\mathcal{E}_{t,t}[X] = X$;
(**A3**) $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X]$;
(**A4**) $1_A \mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[1_A X],\ \ \forall A \in \mathcal{F}_s$.
where $1_A(\omega)$ is the indicator of A

$$1_A = \begin{cases} 1, & \omega \in A; \\ 0, & \omega \notin A. \end{cases}$$

Interpretation:
♠ The meaning of (A1) and (A2) are obvious.
♠ In (A3): $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X],\ r \le s \le t$
at the time r, the value $\mathcal{E}_{s,t}[X]$ is also regarded as a risky payoff with the maturity s. The price of this risky payoff $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]]$ is the same as the price of the original derivative X with maturity t, i.e., $\mathcal{E}_{r,t}[X]$.
♠ In (A4): $1_A \mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[1_A X],\ \forall A \in \mathcal{F}_s$
1_A is considered as a “digital option”. (A4) means that, at time s, the agent knows whether 1_A worths 1 or zero.
If $1_A = 1$, then the value $\mathcal{E}_{s,t}[1_A X]$ is the same as $\mathcal{E}_{s,t}[X]$ since the two outcomes X and $1_A X$ are exactly the same. Otherwise it costs zero.

Definition. A family of mappings $\{\mathcal{E}_{s,t}[\cdot]\}_{0\le s\le t<\infty}$ satisfying (A1)-(A4) is called an $\mathcal{F}_t$–consistent evaluation.

1.2. A particular situation: $\mathcal{F}$–consistent nonlinear expectation

If in the place of (A2) we make a more strong condition:
(**A2′**) $\mathcal{E}_{t,T}[X] = X, \ \ \forall t \in [0,T], \ \forall X \in m\mathcal{F}_t$
We define: $\mathcal{E}[X|\mathcal{F}_t] := \mathcal{E}_{t,T}[X], \ \ \mathcal{E}[X] := \mathcal{E}[X|\mathcal{F}_0] = \mathcal{E}_{0,T}[X]$
We have, $\forall X \in m\mathcal{F}_T, \ s \le t \le T$
$\mathcal{E}[\mathcal{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}_{s,t}[\mathcal{E}_{t,T}[X]] = \mathcal{E}_{s,T}[X] = \mathcal{E}[X|\mathcal{F}_s]$
$\mathcal{E}[\mathcal{E}[X|\mathcal{F}_s]|\mathcal{F}_t] = \mathcal{E}_{t,T}[\mathcal{E}_{s,T}[X]] = \mathcal{E}_{s,T}[X] = \mathcal{E}[X|\mathcal{F}_s]$
In particular, $\forall t \le T, \ A \in \mathcal{F}_t, \ X \in m\mathcal{F}_T$

$$\begin{aligned}\mathcal{E}[X1_A] &= \mathcal{E}_{0,t}[\mathcal{E}_{t,T}[X1_A]] \\ &= \mathcal{E}_{0,t}[\mathcal{E}_{t,T}[X]1_A] \\ &= \mathcal{E}_{0,t}[\mathcal{E}_{t,T}[\mathcal{E}_{t,T}[X]1_A]] \\ &= \mathcal{E}_{0,T}[\mathcal{E}_{t,T}[X]1_A]\end{aligned}$$

i.e.,

$$\mathcal{E}[X1_A] = \mathcal{E}[\mathcal{E}[X|\mathcal{F}_t]1_A], \ \ \forall A \in \mathcal{F}_t.$$

We call $\mathcal{E}[\cdot] : m\mathcal{F}_T \to \mathbf{R}$: an $\mathcal{F}_t$–consistent nonlinear expectation.

2. $\mathcal{F}$–Consistent Evaluation by BSDE

A large kind of $\mathcal{F}$–consistent evaluation can be derives via BSDE

♢ $(\Omega, \mathcal{F}, P)$: A probability space

♢ B_t, $t \in [0,T]$: a d–dimensional Brownian motion on $[0,T]$

$$\mathcal{F}_t := \sigma\{B_s, \ 0 \le s \le t\}.$$

♢ $L^2(\mathcal{F}_t)$ the collection of $\mathcal{F}_t$–measurable random variables such that

$$E[X^2] < \infty.$$

♢ $L^2_{\mathcal{F}}(0,t;R^m)$: all R^m–valued and $\{\mathcal{F}_s\}_{s\ge 0}$–adapted stochastic processes such that

$$E\int_0^t |\phi_s|^2 ds < \infty$$

$\diamondsuit$ $D^2_{\mathcal{F}}(0,t)$: RCLL processes in $L^2_{\mathcal{F}}(0,t) = L^2_{\mathcal{F}}(0,t;R)$ such that

$$E[\sup_{0\le s\le t} |\phi_s|^2] < \infty.$$

$\diamondsuit$ $S^2_{\mathcal{F}}(0,t)$: processes in $D^2_{\mathcal{F}}(0,t)$ with continuous paths.

We consider an $\mathcal{F}_t$–consistent evaluation:

$$\mathcal{E}_{s,t}[\cdot] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s),\ 0 \le s \le t \le T.$$

Our axiomatic assumptions are

for each $0 \le r \le s \le t$, for each $X,\ X' \in L^2(\mathcal{F}_t)$,

(**A1**) $\mathcal{E}_{s,t}[X] \ge \mathcal{E}_{s,t}[X'],\ if\ \ X \ge X'$;
(**A2**) $\mathcal{E}_{t,t}[X] = X$;
(**A3**) $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X]$;
(**A4**) $1_A\mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[1_A X],\ \ \forall A \in \mathcal{F}_s$.

A special situation: when (A2) is replaced by a more strong condition:
(**A2′**) $\mathcal{E}_{s,t}[X] = X,\ \forall X \in L^2(\mathcal{F}_s),\ 0 \le s \le t$.
Interpretation: zero interest rate.
We set $\mathcal{E}[X|\mathcal{F}_t] := \mathcal{E}_{t,T}[X]$

3. g–Evaluation and g–expectations

For a given $X \in L^2(\mathcal{F}_t)$, we solve the following BSDE:

$$Y_s = X + \int_s^t g(r, Y_r, Z_r)dr - \int_s^t Z_r dB_r, \ s \leq t. \qquad \text{(BSDE)}$$

Here the function

$$g(\omega, t, y, z) : \Omega \times [0, T] \times R \times R^d \to R$$

g satisfies condition

$$\begin{cases} \text{(i)} \quad g(\cdot, y, z) \in L^2_{\mathcal{F}}(0, T), \ \ g(t, 0, 0) \equiv 0; \\ \text{(ii)} \quad |g(t, y, z) - g(t, y', z')| \leq \mu(|y - y'| + |z - z'|), \\ \qquad \forall y, y' \in R, \ \ z, z' \in R^d \end{cases} \qquad \text{(g)}$$

Theorem. We assume (g). Then there exists a unique pair

$$(Y, Z) = (Y^{t,X}, Z^{t,X}) \in S^2_{\mathcal{F}}(0, t) \times L^2_{\mathcal{F}}(0, t; R^d)$$

solution of (BSDE).

Remark. The Lipschitz condition in (g) can be generated to the case where g is continuous in y, z and

(a) $|g(t, y, z)| \leq \mu(|y| + |z|), \ \forall (y, z) \in R \times R^d,$

or

(b) $|g(t, y, z)| \leq \mu(1 + |y| + |z|^2), \ \forall (y, z) \in R \times R^d.$

(see Kobylanski, and San Martin & Lepeltier). If g is only continuous in (y, z), there is no uniqueness. We should consider the smallest or the largest solution.

Definition. We set

$$\mathcal{E}^g_{s,t}[X] := Y^{t,X}_s.$$

The system of operators

$$\mathcal{E}^g_{s,t}[X] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \quad 0 \le s \le t.$$

is called g–evaluation.

It is a typical dynamic pricing mechanism!

Theorem. Assume that the function g satisfies (g). Then

$$\{\mathcal{E}^g_{s,t}[\cdot]\}_{0\le s\le t\le T}$$

is an $(\mathcal{F}_t)_{t\ge 0}$–consistent nonlinear evaluation, i.e. it satisfies:

for each $0 \le r \le s \le t$ and for each X, $X' \in m\mathcal{F}_t$,

(**A1**) $\mathcal{E}^g_{s,t}[X] \ge \mathcal{E}_{s,t}[X']$, if $X \ge X'$;

(**A2**) $\mathcal{E}^g_{t,t}[X] = X$;

(**A3**) $\mathcal{E}^g_{r,s}[\mathcal{E}^g_{s,t}[X]] = \mathcal{E}^g_{r,t}[X]$;

(**A4**) $1_A\mathcal{E}^g_{s,t}[X] = \mathcal{E}^g_{s,t}[1_A X]$, $\forall A \in \mathcal{F}_s$.

Sketch of proof.

- (A1): the comparison theorem of BSDE ([P1991], [EPQ1997])
- (A2)–(A3) are clear,

- We now prove (A4): $\forall\, u \in [s,t]$, we have

$$1_A Y_u = 1_A X + \int_u^t 1_A g(r, Y_r, Z_r)dr - \int_u^t 1_A Z_r dB_r.$$

Namely

$$1_A Y_u = 1_A X + \int_u^t g(r, 1_A Y_r, 1_A Z_r)dr - \int_u^t 1_A Z_r dB_r. \qquad \square$$

4. *Example: Risk measure of contingent claims*

Let $X \in L^2(\mathcal{F}_T)$ be a contingent claim ($X \geq 0$) with maturity T, written at the time $t < T$ in a financial market. At the time T, the market must pay the buyer X given from the writer. The minimum cash deposited in the market at the present time t is denoted by $\mathcal{E}_{t,T}[X]$. This is a mapping

$$\mathcal{E}_{t,T}[\cdot] : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t).$$

- A safty but bad policy is $\mathcal{E}_{t,T}[X] = \text{esssup}_\omega X(\omega)$.
- **An ideal policy is** the replicating cost of X: $\mathcal{E}^{r,\theta}_{t,T}[X] := Y^{r,\theta}_t$,

Solution the BSDE

$$-\, dY^{r,\theta}_s = [-r(s)Y^{r,\theta}_s - \theta(s)\cdot Z^{r,\theta}_s]ds - Z^{r,\theta}_s dB_s, \ t \leq s \leq T,$$
$$Y^{r,\theta}_T = X$$

where $\theta(t) = \sigma(t)^{-1}(b(t) - r(t))$.

A big problem: at time t, we don't know $r(s)$ and $\theta(s)$, $t \leq s \leq T$. Usually we only know a range:

$$\mathcal{K} = \{(r(\cdot), \theta(\cdot)) \in L^2_{\mathcal{F}}(0,T) : (r(s), \theta(s)) \in K, \quad \forall s\}$$

where $K \subset R^{d+1}$ is given.

A wise Solution: define

$$g^K(y,z) = \max_{(r,\theta)\in K}[-ry - \theta \cdot z]$$

and then $\mathcal{E}^{g^K}_{t,T}[X] := Y_t$, solution of the nonlinear BSDE:

$$-dY_s = g^K(Y_s, Z_s)ds - Z_s dB_s,\ 0 \le s \le T,$$
$$Y_T = X.$$

We can prove that ([EPQ1997]),

(i) $\mathcal{E}^{g^K}_{t,T}[X] \ge Y^{r,\theta}_t, \quad \forall (r(\cdot), \theta(\cdot)) \in \mathcal{K}$;

(ii) there exists $(r^*(\cdot), \theta^*(\cdot)) \in \mathcal{K}$ such that, for each $t \le T$,

$$\mathcal{E}^{g^K}_{t,T}[X] = Y^{r^*,\theta^*}_t.$$

- $\mathcal{E}^{g^K}_{t,T}[X]$ is also the price given most conservative writer at the time t.
- Observe the price of the most conservative buyer is $\mathcal{E}^{g_K}_{t,T}[X]$

$$g_K(y,z) = \min_{(r,\theta)\in K}[-ry - \theta \cdot z]$$

We have

$\mathcal{E}^{g^K}_{t,T}[X] \ge \mathcal{E}^{g_K}_{t,T}[X]$. If $X \not\equiv EX$, then $P(\mathcal{E}^{g^K}_{t,T}[X] > \mathcal{E}^{g_K}_{t,T}[X]) > 0$.

4.1. *A Special case: nonlinear expectations*

If we assume furthermore that

$$g(t,y,0) \equiv 0 \qquad \text{(g0)}$$

Then, for each $0 \le s \le t$

$$\mathcal{E}^g_{s,t}[X] = X,\ \ \forall X \in L^2(\mathcal{F}_s).$$

Indeed, the pair of processes $(Y_u^{t,X}, Z_u^{t,X}) \equiv (X, 0),\ u \in [s,t]$ solves

$$Y_s^{t,X} = X + \int_s^t g(r, Y_r^{t,X}, Z_r^{t,X})dr - \int_s^t Z_r^{t,X} dB_r,\ s \le t.$$

Definition. Assume (g0). For each $X \in L^2(\mathcal{F}_T)$ and $t \le T$, we set

$$\mathcal{E}^g[X|\mathcal{F}_s] := \mathcal{E}^g_{s,T}[X],$$
$$\mathcal{E}^g[X] := \mathcal{E}^g[X|\mathcal{F}_0] = \mathcal{E}^g_{0,T}[X].$$

- $\mathcal{E}^g[\cdot] : \quad L^2(\mathcal{F}_T) \to R$ is called the g–expectation of X,
- $\mathcal{E}^g[\cdot|\mathcal{F}_s] \ : \ L^2(\mathcal{F}_T) \to \ L^2(\mathcal{F}_s)$ is called the conditional g-expectation of X under $\mathcal{F}_s$.

(A1)–(A4) become:

(**A1**) If $X \ge X'$, a.s. then $\mathcal{E}^g[X] \ge \mathcal{E}^g[X']$ and $\mathcal{E}^g[X|\mathcal{F}_t] \ge \mathcal{E}^g[X'|\mathcal{F}_t]$;

(**A2**) $\mathcal{E}^g[c] = c$, more generally $\mathcal{E}^g[X|\mathcal{F}_t] = X$, $\forall X \in L^2(\mathcal{F}_t)$;

(**A3**) $\mathcal{E}^g[\mathcal{E}^g[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}^g[X|\mathcal{F}_{t\wedge s}]$;

(**A4**) $\mathcal{E}^g[X1_A] = 1_A\mathcal{E}^g[X|\mathcal{F}_t]$, $\forall A \in \mathcal{F}_t$.

Remark. For each $X \in L^2(\mathcal{F}_T)$, $\mathcal{E}^g[X|\mathcal{F}_t]$ is the unniqe r.v. in $L^2(\mathcal{F}_t)$ s.t.

$$\mathcal{E}^g[X1_A] = \mathcal{E}^g[1_A\mathcal{E}^g[X|\mathcal{F}_t]],\ \ \forall A \in \mathcal{F}_t.$$

- This formula uniquely defines the conditional g–expectation under $\mathcal{F}_t$!!!

4.2. Many properties in classical stochastics still holds!!

Definition. A process $Y \in D^2_{\mathcal{F}}(0,T)$ is called

a g–martingale if $\mathcal{E}^g[Y_t|\mathcal{F}_s] = Y_s$;
a g–supermartingale if $\mathcal{E}^g[Y_t|\mathcal{F}_s] \le Y_s$;
a g–submartingale if $\mathcal{E}^g[Y_t|\mathcal{F}_s] \ge Y_s$;
$\forall 0 \le s \le t \le T$

In general,

Definition. $Y \in D^2_{\mathcal{F}}(0,T)$ is called

a g–martingale if $\mathcal{E}^g_{s,t}[Y_t] = Y_s$;
a g–supermartingale if $\mathcal{E}^g_{s,t}[Y_t] \le Y_s$;
a g–submartingale) if $\mathcal{E}^g_{s,t}[Y_t] \ge Y_s$;
$\forall 0 \le s \le t \le T$

Theorem. (Nonlinear Decomposition Theorem of g–supermartingale). If $g(\omega,t,0) \equiv 0$, $\forall(\omega,t,y)$. Let $Y \in D^2_{\mathcal{F}}(0,T)$ be a g–supermartingale. Then there exists a unique increasing process $A \in D^2_{\mathcal{F}}(0,T)$ such that $Y + A$ is a g–martingale:

$$\mathcal{E}^g[Y_t + A_t|\mathcal{F}_s] = Y_s + A_s, \ \ \forall 0 \le s \le t.$$

General case (without $g(\omega,t,y,0) \equiv 0$) For each $X \in L^2(\mathcal{F}_t)$ and $K \in D^2_{\mathcal{F}}(0,T)$ we consider the BSDE

$$Y^{t,X,K}_s = X + K_t - K_s + \int_s^t g(r, Y^{t,X,K}_r, Z^{t,X,K}_r)dr - \int_s^t Z^{t,X,K}_r dB_r, \ s \le t.$$

and set $\mathcal{E}^g_{s,t}[X;K] := Y^{t,X,K}_s$.

(**A1**) $\mathcal{E}^g_{s,t}[X;K] \ge \mathcal{E}_{s,t}[X';K]$, if $X \ge X'$;

(**A2**) $\mathcal{E}^g_{t,t}[X;K] = X$;

(**A3**) $\mathcal{E}^g_{r,s}[\mathcal{E}_{s,t}[X;K];K] = \mathcal{E}_{r,t}[X;K]$;

(**A4$'$**) $1_A\mathcal{E}^g_{s,t}[X;K] = 1_A\mathcal{E}^g_{s,t}[1_AX;K]$, $\forall A \in \mathcal{F}_s$.

Theorem (Nonlinear Decomposition Theorem of g–supermartingale [P1999])

Let $Y \in D^2_{\mathcal{F}}(0,T)$ be a g–supermartingale. Then there exists a unique increasing process $A \in D^2_{\mathcal{F}}(0,T)$ such that

$$\mathcal{E}^g_{s,t}[Y_t; A] = Y_s, \ \ \forall 0 \leq s \leq t.$$

Sketch of Proof.

Penalization approach (introduced in [E-K-P-P-Q 1997])

$$\mathcal{E}^g_{t,T}[Y^{(n)}_t; n\int_0^{\cdot} (Y^{(n)}_r - Y_r)^- dr] = Y^{(n)}_t, \ \ t \leq T, \qquad (Yn)$$

The key point: we can prove that $Y^{(n)} \leq Y$. Thus by comparison theorem

$$Y^n \nearrow Y.$$

We also have

$$A^{(n)} \ \ : \ \ = n\int_0^{\cdot} (Y^{(n)}_r - Y_r)^- dr \ \rightharpoonup A \ \ \text{(weakly in L}^2\text{)}.$$

By a technique introduced in [Peng 1999] (monotonicity limit theorem, we can pass the limit in $(Y^{(n)})$.

$$\mathcal{E}^g_{t,T}[Y_t; A] = Y_t$$

□

5. $\mathcal{F}_t$–evaluation determined by a function g

A more interesting problem: given $\mathcal{F}_t$–consistent evaluation $\mathcal{E}[\cdot]$, is it a g–evaluation?

In general, this is not true. We oberve an key fact: $\mathcal{E}^g_{s,t}[\cdot]$ is dominated by $\mathcal{E}^{g_\mu}_{s,t}[\cdot]$ in the following sense:

$$\mathcal{E}^g_{s,t}[X] - \mathcal{E}^g_{s,t}[Y] \leq \mathcal{E}^{g_\mu}_{s,t}[X - Y], \quad \forall s \leq t, \ \ \forall X, Y \in L^2(\mathcal{F}_t).$$

where $g^\mu(y, z) := \mu|y| + \mu|z|$.

Thus we ask the following question: If an $\mathcal{F}_t$–consistent nonlinear evaluation is dominated by $\mathcal{E}^{g_\mu}[\cdot]$: can we find a function g such that $\mathcal{E}[\cdot] \equiv \mathcal{E}^g[\cdot]$?

Theorem A. ([Peng, 2003]) Let $\{\mathcal{E}_{s,t}[\cdot]\}_{0\leq s\leq t}$ be an $\mathcal{F}_t$–consistent evaluation. If it is dominated by $\mathcal{E}^{g_\mu}[\cdot]$

$$\mathcal{E}_{s,t}[X] - \mathcal{E}_{s,t}[Y] \leq \mathcal{E}^{g_\mu}_{s,t}[X - Y], \quad \forall X, Y \in L^2(\mathcal{F}_t). \qquad \text{(H2)}$$

Then there exists a unique function $g(\omega, t, y, z)$ satisfying (Lip) and $g(\cdot, 0, 0) \equiv 0$ such that,

$$\mathcal{E}^g_{s,t}[X] = \mathcal{E}_{s,t}[X], \ \forall s \leq t, \ \forall X \in L^2(\mathcal{F}_t).$$

A special case of this result was obtained in [C-H-M-P,2002]

6. ***Behaviors of*** g–***evaluations***

Recent results

Proposition. Let g, $\bar{g}$ satisfy (Lip). Then

(i) $g(t, y, z) \geq \bar{g}(t, y, z)$, for each $(y, z) \in R \times R^d$, for a.e. $t \in [0, T]$, a.s.

$\Updownarrow$

(ii) $\forall\ 0 \le s \le t$, $\forall X \in L^2(\mathcal{F}_t)$

$$\mathcal{E}^g_{s,t}[X] \ge \mathcal{E}^{\bar{g}}_{s,t}[X].$$

Proposition. We have

(i) $g(t,y,z)$ is concave (resp. convex) in (y,z),

$\Updownarrow$

(ii) $\mathcal{E}^g_{s,t}[X]$ is concave (resp. convex) in X.

Proposition. We have

(i) $\mathcal{E}^g_{s,t}$ is positively homogenous: $\mathcal{E}^g_{s,t}[\lambda X] = \lambda \mathcal{E}^g_{s,t}[X]$, $\forall\ \lambda \ge 0$;

$\Updownarrow$

(ii) g is positively homogenous: $g(t,\lambda y,\lambda z) = \lambda g(t,y,z)$, $\forall\ \lambda \ge 0$.

Proposition. We have

(i) $\mathcal{E}^g_{s,t}$ is subadditive, i.e. $\mathcal{E}^g_{s,t}[X+X'] \le \mathcal{E}^g_{s,t}[X] + \mathcal{E}^g_{s,t}[X']$;

$\Updownarrow$

(ii) g is subadditive in (y,z), i.e., for almost all $t \in [0,T]$.

Similarly $\mathcal{E}^g_{s,t}[\cdot]$ is superadditive (resp. sublinear, superlinear, linear)

$\Updownarrow$

g is superadditive (resp. sublinear, superlinear, linear).

Proposition. We have

(i) g is independent of y;

$\Updownarrow$

(ii) $\mathcal{E}^g_{s,t}[X+\eta] = \mathcal{E}^g_{s,t}[X] + \eta$, $\forall X \in L^2(\mathcal{F}_t), \eta \in L^2(\mathcal{F}_s)$.

Prop. (i) $\mathcal{E}^g_{s,t}[\cdot]$ *satisfies the* self–financing condition*:* $\mathcal{E}^g_{s,t}[0] \equiv 0$;

$\Updownarrow$

(ii) $g(t,0,0) \equiv 0$.

Zero–interesting rate condition:

Prop. (i) $\mathcal{E}^g_{s,t}[\cdot]$ *satisfies* $\mathcal{E}^g_{s,t}[\eta] = \eta$, $\forall\ \eta \in L^2(\mathcal{F}_s)$;

$\Updownarrow$

(ii) $g(t,y,0) = 0$, $\forall t$ *and* y.

Prop. (i) *For each* $\bar{z}^{i_0}_{\cdot} \in L^2_{\mathcal{F}}(0,T)$

$$\mathcal{E}_{t,T}[X] + \int_0^t \bar{z}_s^{i_0} dB_s^{i_0} = \mathcal{E}_{t,T}[X + \int_t^T \bar{z}_s^{i_0} dB_s^{i_0}]$$

$\Updownarrow$

(ii) *$g(s, y, z)$ does not depends on the i_0th component z^{i_0} of $z \in R^d$.*

7. How to find $g(\omega, t, y, z)$ through the black box $\mathcal{E}^g[\cdot]$

7.1. Non–parameter cases

General case

$$X_s = x + \int_t^s b(X_r^{t,x})dr + \int_t^s a(X_r^{t,x})dB_r, \quad s \geq t.$$

Prop. ([BCHMP], 2003) For each $(t, x, p, y) \in [0, \infty) \times R^n \times R^n \times R$, we have

$$L^2\text{–}\lim_{\epsilon \to 0} \frac{1}{\epsilon}[\mathcal{E}_{t,t+\epsilon}^g[y + p \cdot (X_{t+\epsilon}^{t,x} - x)] - y] = g(t, y, a^T(x)p) + p \cdot b(x).$$

In practice

$$g(t, y, a^T(x)p) \approx \ \{\mathcal{E}_{t,t+\epsilon}^g[y + p \cdot (X_{t+\epsilon}^{t,x} - x)] - y\}\frac{1}{\epsilon} \ - p \cdot b(x)$$

$\log P(t)$

Markovian Properties: An $\mathcal{F}_t$–progressively meas. process $(X_t)_{t\geq 0}$ **is said to be Markovian under $\mathcal{E}[\cdot]$ if for each $s \leq t$ and $\Phi \in C_b(R^n)$, we have**

$$\mathcal{E}[\Phi(X_t)|\mathcal{F}_s] \text{ is } \sigma\{X_s\}\text{–measurable.}$$

Example. Assume $g = g_0(X_t, y, z) + q_t \cdot z,\ q \in L^\infty_{\mathcal{F}}(0,T;R^d)$. Then

$$dX_t = (b(X_t) - q_t)dt + dW_t,\ X_0 = x \in R^n$$

is an $\mathcal{E}^g$–Markovian process.

Sketch of Proof of Theorem A.

We need:

Theorem (Nonlinear Decomposition Theorem of $\mathcal{E}$–supermartingale [Peng 2003])**.** *Let* $Y \in D^2_{\mathcal{F}}(0,T)$ *be a* $\mathcal{E}$*–supermartingale dominated by* $\mathcal{E}^{g_\mu}[\cdot]$*. Then there exists a unique increasing process* $A \in D^2_{\mathcal{F}}(0,T)$ *such that*

$$\mathcal{E}_{s,t}[Y_t; A] = Y_s, \quad \forall 0 \le s \le t.$$

Step 1. Since $Y_s^{t,X} := \mathcal{E}_{s,t}[X]$ is a g_μ–submartingale and $-g_\mu$–supermartingale, there exists an increasing process $A^{t,X} \in D^2_{\mathcal{F}}(0,T)$ and $\bar{A}^{t,X} \in D^2_{\mathcal{F}}(0,T)$, such that

$$Y_s^{t,X} = \mathcal{E}^{g_\mu}_{s,t}[X; -A^{t,X}] = \mathcal{E}^{-g_\mu}_{s,t}[X; A].$$

i.e.,

$$\begin{aligned} -dY_s^{t,X} &= g_\mu(Y_s^{t,X}, Z_s^{t,X})ds - dA_s^{t,X} - Z_s^{t,X}dB_s, \\ -dY_s^{t,X} &= -g_\mu(Y_s^{t,X}, \bar{Z}_s^{t,X})ds + d\bar{A}_s^{t,X} - \bar{Z}_s^{t,X}dB_s. \end{aligned}$$

It then follows that

$$Z_s^{t,X} \equiv \bar{Z}_s^{t,X},\ d(A_s^{t,X} + A_s^{t,X}) \equiv g_\mu(Y_s^{t,X}, Z_s^{t,X})ds.$$

Thus there exists $g^{t,X} \in L^2_{\mathcal{F}}(0,T)$ such that

$$-dY_s^{t,X} = g_s^{t,X} ds - Z_s^{t,X} dB_s,$$

and

$$|g_s^{t,X}| \leq \mu(|Y_s^{t,X}| + |Z_s^{t,X}|).$$

Step 2. Since $Y_s^{t,X} - Y_s^{t,X'}$ is dominated by $\mathcal{E}^{g_\mu}_{s,t}[X - X']$, using the similar argument,

$$|g_s^{t,X} - g_s^{t,X'}| \leq \mu(|Y_s^{t,X} - Y_s^{t,X'}| + |Z_s^{t,X} - Z_s^{t,X'}|).$$

Step 3. For each fixed $(y,z) \in R^{1+d}$, consider a forward SDE

$$\begin{aligned} -dY_s^{t_0,y,z} &= g_\mu(Y_s^{t_0,y,z}, z)ds - zdB_s, \ s \geq t_0, \\ Y_{t_0}^{t_0,y,z} &= y. \end{aligned}$$

Since $Y_s^{t_0,y,z}$ is a $\mathcal{E}$–supermartingale

$$Y_s^{t_0,y,z} = \mathcal{E}_{s,t}[Y_s^{t_0,y,z}; A^{t_0,y,z}]$$

Thus $Y^{t_0,y,z}$ is a $\mathcal{E}_{s,t}[\cdot; A^{t_0,y,z}]$–martingale

$$\Rightarrow \mathcal{E}^{g_\mu}_{s,t}[\cdot; A^{t_0,y,z}]\text{-submartingale.}$$

We use again g–submartingale decomposition theorem:

$$-dY_s^{t_0,y,z} = g_s^{t_0,y,z} ds - Z_s^{t_0,y,z} dB_s, \quad s \in [t,T].$$

7.2. A special situation: $g = g(z)$

$$\mathcal{E}^g_{t,T}[z(B_T - B_t)] = g(z)(T-t).$$