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THE RANGE OF RANDOM WALK ON TREES AND RELATED TRAPPING PROBLEM

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Abstract. This paper treats with the range of the simple random walk on trees and a related trapping problem. The strong law of large numbers and the central limit theorem for the range, and some asymptotic behaviour for the mean trapping time and survival probability are presented.

§1. Introduction.

Let T_N be the infinite tree with $N+1$ branches emanating from each vertex. Namely, T_N is an infinite connected graph with no non-trivial closed loops in which every node belongs to exactly $N+1$ edges. Since T_1 can be thought of as the one dimensional lattice, which is well studied, throughout this paper we assume that $N \geq 2$. Let $\{X_n\}_{n\geq 0}$ be the simple random walk on T_N, with the probability law $\{P_x\}_{x\in T_N}$. The range of $\{X_n\}_{n\geq 0}$ up to time n is denoted by $R_n = \#\{X_0, X_1, \cdots, X_n\}$. Our first purpose is to study the asymptotic behaviour of R_n as $n \to \infty$. For this, the main result is as follows.

Theorem 1.1. Let ζ denote a standard normal variable and let E_x be the expectation with respect to P_x. We have

$$i)\quad \lim_{n\to\infty} R_n/n = (N-1)/N, \qquad P_0 - a.s.,$$

$$ii)\quad \lim_{n\to\infty} var(R_n)/n = (N^2+1)/[N^2(N-1)],$$

$$iii)\quad (R_n - E_0R_n)/n^{1/2} \xrightarrow{(d)} \zeta\,(N^2+1)/[N^2(N-1)], \quad n\to\infty.$$

Next, consider the N-tree $\tilde{T}_N$ with root 0. Each vertex has exactly N − successors. Again, when $N = 1$, $\tilde{T}_N$ can be thought of as the set $\{0, 1, \cdots\}$. We restrict ourselves to the case that $N \geq 2$. Clearly, $\tilde{T}_N$ is a subset of T_N. Our next result concerns with the range of the simple random walk $\{Y_n\}_{n\geq 0}$ on $\tilde{T}_N$. Let $\{P_x\}_{x\in\tilde{T}_N}$ be the probability law and set $\tilde{R}_n = \#\{Y_0, \cdots, Y_n\}$.

Theorem 1.2. The same conclusions of Theorem 1.1 hold provided R_n, E_x and P_x are replaced by $\tilde{R}_n$, $\tilde{E}_x$ and $\tilde{P}_x$ respectively.

Finally, we study the trapping problem on trees. The problem on lattices has been attracted a lot of attentions, refer to [4] and references within. Given $\epsilon > 0$, let $C(x)$, $x \in T_N$ be *i.i.d.* $\{0,\ 1\}$–valued random variables satisfying $P_C(C(x) = 1) = 1 - P_C(C(x) = 0) = \epsilon$ for all $x \in T_N$, where P_C denotes the probability law of $(C(x), x \in T_N)$. The family $(C(x), x \in T_N)$ is called a random trap field with density ϵ. In general, 1 corresponds to a trap, and 0 to a trap–free site. The trapping time and the survival probability are defined by $T = \inf\{n \geq 0 :\ C(X_n) = 1\}$ and $f(n) = P(T > n)$, $n \geq 0$ respectively, where $P = P_0 \times P_C$. In this part, we are interested in asymptotic behaviour of $f(n)$ and E_0T. The main result is as follows.

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Theorem 1.3. i) For small $\epsilon > 0$ and moderate n, we have $\log f(n) \sim -\epsilon(N-1)n/N$.
ii) Let E be the expectation with respect to P, then $\lim_{\epsilon\to 0^+}(\epsilon ET) = N/(N-1)$.

The solution to the trapping problem on $\tilde{T}_N$ is completely the same (see Corollary 6.4 below).

From the arguments in [3] or [4], one knows that Theorem 1.3 is actually a consequence to Theorem 1.1. Thus, we concentrate our attention mainly on the proofs of Theorem 1.1 and Theorem 1.2. Since T_N has some nice symmetric properties and $\{X_n\}_{n\geq 0}$ is transient, some techniques used in [6] or [5], where the corresponding problem was studied for the lattice case, can be also applied to prove Theorem 1.1. A key to [6] or [5] is some reasonable estimate for the Green function and hitting time of random walks on lattices. This was obtained by using some estimate of their transition probability function. Although there are a lot of works in estimating the transition probability function of random walk on trees (e.g. [2] and [7]), it is still difficult to use these estimates to get a reasonable estimate for the corresponding Green function and hitting time. This problem is overcome in the paper in terms of some techniques in electrical network. Besides, in the present case we can get the precise limits as described in Theorem 1.1 and Theorem 1.2. However, the coefficients of the corresponding limits in the lattice case are still not known precisely.

Let us mention that it is also meaningful to study the range of random walks on fractals (see [4]). However, we do not know at moment how to get a precise asymptotic behaviour even for the range of the simple random walk on the Sierpinski gaskets.

This paper is organized as follows. In Section 2, we mainly study the hitting time of $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$. In Section 3, we obtain an asymptotic behaviour of $var(R_n)$ as $n \to \infty$. In Section 4, we prove both strong law of large number and central limit theorem for R_n, and complete the proof Theorem 1.1. In Section 5, we prove that $\tilde{R}_n$ is very close to R_n in some sense (see Lemma 5.1 and Lemma 5.2 below). Thus, Theorem 1.2 can be easily proved by means of Theorem 1.1. In Section 6, we consider the trapping problem on T_N and $\tilde{T}_N$, and complete the proof of Theorem 1.3.

§2. Hitting time.

In this section, we make some reasonable estimation for the mean and variance of the hitting time of $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$. For this purpose, we introduce some notation. For any $x, y \in G$ $(= T_N$, or $\tilde{T}_N)$, define their distance as follows:

$$d(x,y) = \inf\big\{k : \exists x_1, \cdots, x_k \in G, \text{ such that } x_1 = x,\ x_k = y \text{ and } x_i x_{i+1} \text{ is an edge of } G \text{ for } \forall\ i = 1, \cdots, k-1\big\}.$$

Next, define

$$B_n(x) = \{y \in G : d(x,y) \leq n\}, \quad S_n(x) = \{y \in G : d(x,y) = n\},$$
$$\tau_n(x) = \inf\{m \geq 0 : X_m \in S_n(x)\}, \quad \tilde{\tau}_n(x) = \inf\{m \geq 0 : Y_m \in S_n(x)\}.$$

Let 0 be the root of $\tilde{T}_N$, then $0 \in T_N$. For simplicity, set $B_n = B_n(0)$; $S_n = S_n(0)$; $\tau_n = \tau_n(0)$; $\tilde{\tau}_n = \tilde{\tau}_n(0)$.

One of the main results in this section is as follows.

Proposition 2.1. i) $E_0\tau_n = n(N+1)/(N-1) + O(1)$ as $n \to \infty$.
ii) There is a constant $M \in (0,\infty)$ such that $E_0(\tau_n - E_0\tau_n)^2 \leq Mn$ for all $n \geq 1$.

To prove this proposition, construct a random walk $\{Z_n\}_{n\geq 0}$ on $\mathbf{Z}_+ = \{0, 1, \cdots\}$ with transition probability: $p_{ij} = 1$, if $i = 0$ and $j = 1$; $= N/(N+1)$ if $i \geq 1$ and $j = i+1$; $= 1/(N+1)$, if $i \geq 1$ and $j = i-1$ and $= 0$, otherwise. Denote by W_x and Q_x respectively the probability law and its corresponding expectation of $\{Z_n\}$ starting from $x \in \mathbf{Z}_+$. Let $\sigma_n = \inf\{m \geq 0 : Z_m = n\}$, $n \geq 0$. Then, it is easy to see that $W_0(\sigma_n = m) = P_0(\tau_n = m)$, $n \geq 0$, $m \geq 0$. From this, it is clear that Proposition 2.1 follows from the next lemma.

Lemma 2.2. i) $Q_0\sigma_n = n(N+1)/(N-1) + O(1)$ as $n \to \infty$.

ii) There is a constant $M \in (0,\infty)$ such that $Q_0(\sigma_n - Q_0\sigma_n)^2 \le Mn$ for all $n \ge 0$.

Proof. The proof of this lemma is based on an electrical network. To do so, let $C_{i,i+1} = 1$, if $i = 0$ and $= N^{i+1}/(N+1)$, if $i \ge 1$. Consider the electrical network $\mathbf{Z}_+$ in which a conductor $C_{i,i+1}$ is assigned to the bond $\overline{i, i+1}$. Then the effective resistance $R_{\rm eff}(n)$ between 0 and n is equal to

$$R_{\rm eff}(n) = 1 + \sum_{i=1}^{n-1} \frac{N+1}{N^{i+1}} = 1 + \frac{N+1}{N(N-1)}(1 - N^{-n+1}).$$

For given $n \ge 1$, let $u_k = Q_0\Big[\sum_{j=0}^{\sigma_n} I_{\{Z_j = k\}}\Big]$, $0 \le k \le n$ and $v_k = W_k(\sigma_0 < \sigma_n)$, $0 \le k \le n$. Then, we have $Q_0\sigma_n = \sum_{k=0}^{n-1} u_k$. From [1], we know that $u_0 = R_{\rm eff}(n)$ and $u_k/N^k = R_{\rm eff}(n)\, v_k$, $k = 0, 1, \cdots, n$. Therefore, $Q_0\sigma_n = R_{\rm eff}(n) \sum_{k=0}^{n-1} N^k v_k$. Clearly, $v_k's$ are voltages in the electrical network $\mathbf{Z}_+$ between 0 and n having the property $v_0 = 1$ and $v_n = 0$ (see [1]). Thus. if we denote by $R_{\rm eff}(k,n)$ is the effective resistance between k and n, then $v_k = R_{\rm eff}^{-1}(n)\, R_{\rm eff}(k,n)$, $0 \le k \le n$. Hence

$$R_{\rm eff}(k,n) = \sum_{i=k}^{n-1} \frac{N+1}{N^{i+1}} = \frac{N+1}{N^k(N-1)}(1 - N^{-(n-k)}), \quad 1 \le k \le n-1.$$

Collecting the above facts together, we obtain

$$\begin{aligned}
Q_0\sigma_n &= \left[1 + \frac{N+1}{N(N-1)}(1 - N^{-n+1})\right] \\
&\quad \times \left\{1 + \sum_{k=1}^{n-1} \left[1 + \frac{N+1}{N(N-1)}(1 - N^{-n+1})\right]^{-1} \frac{N+1}{N-1}(1 - N^{-(n-k)})\right\} \\
&= \frac{N+1}{N-1}(n-1) - \frac{N+1}{(N-1)^2}(1 - N^{-n+1}) + 1 + \frac{N+1}{N(N-1)}(1 - N^{-n+1}),
\end{aligned}$$

which proves i).

Next, consider $Q_0\sigma_n^2$. By definition, we know that

$$\begin{aligned}
Q_0\sigma_n^2 &= 2 \sum_{0 \le k_1 < k_2 \le n-1} Q_0\big[\#\{j \le \sigma_n :\ Z_j = k_1\} \cdot \#\{j \le \sigma_n :\ Z_j = k_2\}\big] \\
&\quad + 2\sum_{k=0}^{n-1} Q_0\Bigg[\sum_{0 \le i < j \le \sigma_n} I_{\{Z_i = Z_j = k\}}\Bigg] + \sum_{k=0}^{n-1} Q_0\Bigg(\sum_{0 \le i \le \sigma_n} I_{\{Z_i = k\}}\Bigg) \\
&=:\ I_1(n) + I_2(n) + I_3(n).
\end{aligned}$$

From the above argument, one sees that

$$I_3(n) = n(N+1)/(N-1) + O(1), \quad n \to \infty. \tag{2.3}$$

Moreover, by the strong Markov property we also know that

$$\begin{aligned}
Q_0\Bigg(\sum_{0 \le i < j \le \sigma_n} I_{\{Z_i = Z_j = k\}}\Bigg) &= Q_0\Bigg[\sum_{i=0}^{\infty} I_{\{i \le \sigma_n\}} I_{\{Z_i = k\}} Q_k\Bigg(\sum_{j=0}^{\sigma_n} I_{\{Z_j = k\}}\Bigg)\Bigg] \\
&= u_k Q_k\Bigg(\sum_{j=0}^{\sigma_n} I_{\{Z_j = k\}}\Bigg).
\end{aligned} \tag{2.4}$$

Let $\sigma_n' = \inf\{m \ge 0 :\ Z_m \in \{0, n\}\}$. Then

$$Q_k\Bigg(\sum_{j=0}^{\sigma_n} I_{\{Z_j = k\}}\Bigg) \le Q_k\Bigg(\sum_{j=0}^{\sigma_n'} I_{\{Z_j = k\}}\Bigg) + Q_0\Bigg(\sum_{j=0}^{\sigma_n} I_{\{Z_j = k\}}\Bigg). \tag{2.5}$$

Denote by $R'_{\text{eff}}(k,n)$ the effective resistance between k and $\{0,n\}$ $(1 \le k \le n-1)$. It is easy to see that there is a constant $c_1 \in (0,\infty)$ such that $R'_{\text{eff}}(k,n) \le c_1$, $1 \le k \le n$. This implies that $Q_k\Big(\sum_{i=0}^{\sigma'_n} I_{\{Z_i=k\}}\Big) = N^{-k}R'_{\text{eff}}(k,n) \le c_1$, $1 \le k \le n$. Inserting this into (2.5), we obtain $Q_k\Big(\sum_{i=0}^{\sigma_n} I_{\{Z_i=k\}}\Big) \le c_1 + u_k \le c_2$ for some constant $c_2 \in (0,\infty)$. Combining this with (2.4) gives

$$I_2(n) \le c_3 n, \quad \forall n \ge 1 \tag{2.6}$$

for some constant $c_3 \in (0,\infty)$.

Therefore, it remains to prove that there is a constant $c_4 \in (0,\infty)$ such that

$$I_1(n) - (Q_0\sigma_n)^2 \le c_4 n, \quad \forall n \ge 1. \tag{2.7}$$

Indeed, if $k_1 < k_2$, then $W_{k_2}(\sigma_{k_1} < \sigma_n) = R_{\text{eff}}(k_2,n)/R_{\text{eff}}(k_1,n) = N^{k_1-k_2}(1-N^{-(n-k_2)})(1-N^{-(n-k_1)})^{-1}$. Thus, we have

$$\begin{aligned} I_1(n) &= 2\sum_{k_1=0}^{n-1}\sum_{k_2=k_1+1}^{n-1} Q_0\Big\{\big[\#\{j \le \sigma_{k_2} :\ Z_j = k_1\} \cdot Q_{k_2}(\#\{j \le \sigma_n :\ Z_j = k_2\})\big] \\ &\quad + Q_{k_2}\big[\#\{j \le \sigma_n :\ Z_j = k_1\} \cdot \#\{j \le \sigma_n :\ Z_j = k_2\}\big]\Big\} \\ &= 2\sum_{k_1=0}^{n-1}\sum_{k_2=k_1+1}^{n-1} u_{k_2} Q_0(\#\{j \le \sigma_{k_2} :\ Z_j = k_1\}) \\ &\quad + 2Q_{k_2}\big[I_{\{\sigma_{k_1}<\sigma_n\}}\#\{j \le \sigma_n :\ Z_j = k_1\} \cdot \#\{j \le \sigma_n :\ Z_j = k_2\}\big] \\ &\le 2\sum_{k_1=0}^{n-1}\sum_{k_2=k_1+1}^{n-1} \Big[u_{k_2} Q_0(\#\{j \le \sigma_{k_2} :\ Z_j = k_1\}) \\ &\quad + 2Q_{k_2}^{1/2}\big(I_{\{\sigma_{k_1}<\sigma_n\}}\#\{j \le \sigma_n :\ Z_j = k_1\}\big)^2 Q_{k_2}^{1/2}(\#\{j \le \sigma_n :\ Z_j = k_2\})^2\Big]. \end{aligned} \tag{2.8}$$

From the proofs of (2.3) and (2.6), we see that $Q_{k_2}(\#\{j \le \sigma_n :\ Z_j = k_2\})^2 \le c_5$, $k_2 \le n-1$ for some constant $c_5 \in (0,\infty)$. In addition, by the strong Markov property, we have $Q_{k_2}\big[I_{\{\sigma_{k_1}<\sigma_n\}}(\#\{j \le \sigma_n :\ Z_j = k_1\})^2\big] = W_{k_2}(\sigma_{k_1} < \sigma_n) Q_{k_1}(\#\{j \le \sigma_n :\ Z_j = k_1\})^2$, which implies that

$$\begin{aligned} \text{r.h.s. of (2.8)} &\le 2\sum_{k_1=0}^{n-1}\sum_{k_2=k_1+1}^{n-1} u_{k_2} Q_0(\#\{j \le \sigma_{k_2} :\ Z_j = k_1\}) \\ &\quad + 2c_5\sum_{k_1=0}^{n-1}\sum_{k_2=k_1+1}^{n-1} N^{-(k_2-k_1)/2}(1-N^{-(n-k_2)})^{1/2}(1-N^{-(n-k_1)})^{-1/2}. \end{aligned}$$

Hence, to prove (2.7) it suffices to show that

$$2\sum_{k_1=0}^{n-1}\sum_{k_2=k_1+1}^{n-1} u_{k_2} Q_0(\#\{j \le \sigma_{k_2} :\ Z_j = k_1\}) - 2\sum_{k_1=0}^{n-1}\sum_{k_2=k_1+1}^{n-1} u_{k_2}u_{k_1} \le c_6 n,\ n \ge 1 \tag{2.9}$$

for some constant $c_6 \in (0,\infty)$. Note that $Q_0(\sigma_{k_2} < \sigma_n) = 1$ for $k_2 \le n-1$. Then $Q_0\big(\#\{j \le \sigma_{k_2} :\ Z_j = k_1\}\big) - u_{k_1} \le 0$, which yields that l.h.s. of (2.9) ≤ 0. Hence, (2.9) is true. This proves (2.7).

Combining (2.3) and (2.6) with (2.7), we get the desired result. ∎

We have completed the proof of Proposition 2.1. By a similar argument, we can prove the following result.

Proposition 2.10. i) $\tilde{E}_0\tilde{\tau}_n = n(N+1)/(N-1) + O(1)$, as $n \to \infty$.
ii) There is a constant $M \in (0,\infty)$ such that $\tilde{E}_0(\tilde{\tau}_n - \tilde{E}_0\tilde{\tau}_n)^2 \le Mn$ for all $n \ge 1$.

In fact, from the definitions of $\tilde{\tau}_n$ and σ_n, one can also see that $\tilde{P}_0(\tilde{\tau}_n = m) = Q_0(\sigma_n = m)$, for all $n, m \ge 0$. By using this, Proposition 2.10 follows from Lemma 2.2.

§3. Variance of R_n.

Following [6] (or [5]), let

$$\begin{aligned}
&\xi_n^n = 1; \quad \xi_i^n = I_{\{X_i \ne X_{i+1}, \cdots, X_i \ne X_n\}}, \quad 0 \le i < n; \\
&\xi_i = I_{\{X_i \ne X_{i+1}, X_i \ne X_{i+2}, \cdots\}}, \quad i \ge 0; \\
&\eta_i^n = \xi_i^n - \xi_i, \quad 0 \le i < n; \quad \zeta_n = \textstyle\sum_{i=0}^{n-1} \xi_i; \quad \eta_n = \sum_{i=0}^{n-1} \eta_i^n.
\end{aligned}$$

Then $R_n = \sum_{i=1}^n \xi_i^n = \zeta_n + \eta_n + 1$. Next, let $p_k(x,y) = P_x(X_k = y)$ and set

$$\begin{aligned}
&H_x = \inf\{n \ge 1 : \ X_n = x\}; \quad F(x,y) = P_x(H_y < \infty); \\
&G_{(n)}(x,y) = \textstyle\sum_{k=0}^n p_k(x,y); \quad G(x,y) = G_{(\infty)}(x,y); \\
&p_k^z(x,y) = P_x(X_k = y; \ H_z \ge k).
\end{aligned}$$

The main result in this section is as follows.

Proposition 3.1. Let $\sigma = (N^2+1)/[N^2(N-1)]$, then $\lim_{n\to\infty} var(R_n)/n = \sigma$.

To prove this proposition, we begin with two lemmas.

Lemma 3.2. For any $x, y \in T_N$, we have $G(x,y) = (N+1)^{-d(x,y)+1}(N-1)^{-1}$.

Proof. By the symmetry of T_N, it suffices to prove that $G(0,x) = (N-1)^{-1}(N+1)^{-d(0,x)+1}$ for all $x \in T_N$. Consider the electrical network T_N in which a unit resistor is assigned to each bond of T_N. Let $(v_x)_{x\in T_N}$ be the voltage on T_N satisfying $v_0 = 1$ and $\lim_{d(0,x)\to\infty} v_x = 0$. Denote by R_{eff} the effective resistane of T_N between 0 and infinity. Then (see [1]), we have $G(0,0)/(N+1) = R_{\text{eff}}$ and $G(0,x)/(N+1) = R_{\text{eff}}\, v_x$. It is clear that $R_{\text{eff}} = (N+1)^{-1} + \sum_{n=1}^\infty (N+1)^{-1}N^{-n} = N(N^2-1)^{-1}$ and $v_x = R_{\text{eff}}^{-1} \cdot R_{\text{eff}}(x)$ for all $x \in T_N$, where $R_{\text{eff}}(x)$ is the effective resistance of T_N between $S_{d(0,x)}$ and infinity. One may check that

$$R_{\text{eff}}(x) = \sum_{n=d(0,x)}^\infty \frac{1}{(N+1)N^n} = \frac{1}{(N+1)N^{d(0,x)}} \cdot \frac{N}{N-1} = \frac{1}{N^2-1} N^{-d(0,x)+1}.$$

Therefore, $v_x = N^{-d(0,x)}$ for all $x \in T_N$, which implies that $G(0,x) = (N+1)N(N^2-1)^{-1}N^{-d(0,x)} = (N-1)^{-1}N^{-d(0,x)+1}$. ■

Lemma 3.3. There is a constant $c \in (0,\infty)$ such that

$$\textstyle\sum_{x\in T_N} G_{(n)}(0,x) P_x(m < H_x < \infty, \ H_0 < \infty) \le c\, m^{-3/2}, \quad \forall m \ge 1, \ n \ge 0.$$

In the lattice case, the above bound was obtained by using an estimation of transition probability (see [5] or [6]). Here, we use Lemma 3.2 and Proposition 2.1 to prove Lemma 3.3.

Proof of Lemma 3.3. First, we show that

$$P_y(m < H_z < \infty) \le c_1 m^{-3/2}, \quad \forall m \ge 1, \ \forall y, z \in T_N \tag{3.4}$$

for some constant $c_1 \in (0,\infty)$. Indeed, if $d(y,z) \ge (\log m)^2$, then Lemma 3.2 implies

$$P_y(m < H_z < \infty) \le P_y(H_z < \infty) \le G(y,z) \le (N-1)^{-1}(N+1)^{-(\log m)^2+1}.$$

Thus, (3.4) holds for $d(y,z) \geq (\log m)^2$. We now assume that $d(y,z) \leq (\log m)^2$. By Proposition 2.1, we can show that $P_y(\tau_{[m^{1/4}]}(y) \geq m) \leq c_2 m^{-3/2}$ for some constant $c_2 \in (0,\infty)$. Thus, to prove (3.4) it suffices to show the following:

$$P_y(\tau_{[m^{1/4}]}(y) < H_z < \infty) \leq c_1 m^{-3/2}, \qquad d(y,z) \leq (\log m)^2. \tag{3.5}$$

In fact, by Lemma 3.2 and the strong Markov property, we have

$$\begin{aligned}\text{l.h.s. of (3.5)} &= E_y\big[P_{X(\tau_{[m^{1/4}]}(y))}(H_z < \infty)\big] \leq E_y\big[G(X(\tau_{[m^{1/4}]}(y)), z)\big] \\ &= E_y (N-1)^{-1}(N+1)^{1-d(z,X(\tau_{[m^{1/4}]}(y)))},\end{aligned}$$

where $X(m) = X_m$, $m \geq 0$. Recalling the hypothesis: $d(y,z) \leq (\log m)^2$, we get $d(z, X(\tau_{[m^{1/4}]}(y))) \geq [m^{1/4}] - (\log m)^2 - 1$. This implies (3.5) immediately. Hence, (3.4) holds. Thus, if $x \neq 0$, then

$$\begin{aligned}&P_x(m < H_x < \infty;\ H_0 < \infty) \\ &\leq P_x(m < H_x < \infty)F(x,0) + P_x(m/2 < H_0 < \infty)F(0,x) + F(x,0)P_0(m/2 < H_0 < \infty) \\ &\leq c_2 m^{-3/2}(G(0,x) + G(x,0)), \quad \forall m \geq 1,\ \forall x \in T_N\end{aligned}$$

for some constant $c_2 \in (0,\infty)$. Actually, the above bound also holds for $x = 0$. Therefore,

$$\begin{aligned}&\sum_{x \in T_N} G_{(n)}(0,x) P_x(m < H_x < \infty;\ H_0 < \infty) \\ &\leq 2c_2 m^{-3/2} \sum_{x \in T_N} G^2(0,x) \leq c_3 m^{-3/2}, \qquad \forall m \geq 1,\ \forall n \geq 1\end{aligned}$$

for some constant $c_3 \in (0,\infty)$. ■

We are now in the position to prove Proposition 3.1.

Proof of Proposition 3.1. By the symmetry of T_N, we easily see that for $i < j$,

$$\begin{aligned}E_0 \eta_i^n \eta_j^n &= E_0\big[E_{X_i}(\eta_0^{n-i}\eta_{j-i}^{n-i})\big] \\ &= P_0\big(X_1 \neq 0, \cdots, X_{n-i} \neq 0;\ X_k = 0 \text{ for some } k \geq n-i+1; \\ &\quad X_{j-i} \neq X_{j-i+1}, \cdots, X_{j-i} \neq X_{n-i};\ X_k = X_{j-i} \text{ for some } k \geq n-i+1\big) \\ &= \textstyle\sum_{x \neq 0} p_{j-i}^0(0,x) P_x(n-j+1 \leq H_0 < \infty;\ n-j+1 \leq H_x < \infty) \\ &\leq \textstyle\sum_{x \neq 0} p_{j-i}(0,x) P_x(n-j+1 \leq H_x < \infty;\ H_0 < \infty).\end{aligned}$$

In fact, the above bound is also valid for $i = j$. By Lemma 3.3, we have

$$\sum_{i=0}^{j} E_0(\eta_i^n \eta_j^n) \leq \sum_{x \neq 0} G_{(j)}(0,x) P_x(n-j+1 \leq H_x < \infty;\ H_0 < \infty) \leq c_4 (n-j+1)^{-3/2}, \quad j \leq n$$

for some constant $c_4 \in (0,\infty)$. It follows that $E_0 \eta_n^2 = O(1)$ as $n \to \infty$. By definition, we know that $var(R_n) = var(\zeta_n) + var(\eta_n) + 2cov(\zeta_n, \eta_n)$. Thus, the desired result follows once we prove the following:

$$\lim_{n \to \infty} var(\zeta_n)/n = \sigma. \tag{3.6}$$

We now compute $var(\zeta_n)$. $var(\zeta_n) = \sum_{i=0}^{n-1} var(\xi_i) + 2\sum_{0 \leq i < j \leq n-1} cov(\xi_i, \xi_j)$. By the symmetry of T_N, we also have

$$\begin{aligned}var(\xi_i) &= P_0(X_i \neq X_{i+1},\ X_i \neq X_{i+2}, \cdots) - P_0(X_i \neq X_{i+1},\ X_i \neq X_{i+2}, \cdots)^2 \\ &= P_0(X_1 \neq 0,\ X_2 \neq 0, \cdots) - P_0(X_1 \neq 0,\ X_2 \neq 0, \cdots)^2 = q - q^2,\end{aligned}$$

where $q = P_0(X_1 \neq 0,\ X_2 \neq 0, \cdots) = R_{\rm eff}^{-1}\,(N+1)^{-1} = (N-1)/N$ (see [1]). Moreover, we have

$$\begin{aligned}E_0\xi_i\xi_j &= E_0\xi_0\xi_{j-i} = P_0(X_n \neq 0,\ \forall n \geq 1;\ X_{j-i} \neq X_{j-i+m},\ \forall m \geq 1)\\ &= \textstyle\sum_{x\neq 0} p^0_{j-i}(0,x) P_x(H_x = \infty,\ H_0 = \infty)\\ &= \textstyle\sum_{x\neq 0} p^0_{j-i}(0,x)\big[P_x(H_x = \infty) - P_x(H_x = \infty,\ H_0 < \infty)\big]\\ &= P_0(X_1 \neq 0, \cdots, X_{j-i} \neq 0)P_0(H_0 = \infty) - \textstyle\sum_{x\neq 0} p^0_{j-i}(0,x) P_x(H_x = \infty,\ H_0 < \infty),\end{aligned}$$

and $E_0\xi_i E_0\xi_j = P_0(H_0 = \infty)P_0(X_k \neq 0,\ \forall k \geq 1)$. Therefore

$$\begin{aligned}cov(\xi_i, \xi_j) &= P_0(H_0 = \infty)P_0\big(X_1 \neq 0, \cdots, X_{j-i} \neq 0;\ X_k = 0 \text{ for some } k \geq j-i+1\big)\\ &\qquad - \textstyle\sum_{x\neq 0} p^0_{j-i}(0,x) P_x(H_x = \infty,\ H_0 < \infty)\\ &= \textstyle\sum_{x\neq 0} p^0_{j-i}(0,x)\big[P_x(H_x = \infty)P_x(H_0 < \infty) - P_x(H_x = \infty;\ H_0 < \infty)\big].\end{aligned}$$

Let $a_j = \sum_{i=1}^{j}\sum_{x\neq 0} p^0_i(0,x)\big[P_0(H_0 = \infty)P_x(H_0 < \infty) - P_x(H_x = \infty;\ H_0 < \infty)\big]$. Then $var(\zeta_n) = n(q - q^2) + 2\sum_{j=1}^{n-1} a_j$. Put $a = \sum_{i=1}^{\infty}\sum_{x\neq 0} p^0_i(0,x)\big[P_0(H_0 = \infty)P_x(H_0 < \infty) - P_x(H_x = \infty;\ H_0 < \infty)\big]$. We have $\lim_{n\to\infty} var(\xi_n)/n = q - q^2 + 2a$. Hence, for proving (3.6) it remains to prove the following

$$\sigma = q - q^2 + 2a. \tag{3.7}$$

Note that if $x \neq 0$, $F(0,x) = v_x = N^{-d(o,x)}$, where v_x was defined in Lemma 3.1. Therefore, if $x \neq 0$,

$$\begin{aligned}&P_x(H_x = \infty)P_x(H_0 < \infty) - P_x(H_x = \infty;\ H_0 < \infty)\\ &= P_x(H_x = \infty)P_x(H_0 < \infty) - P_x(H_0 < \infty) + P_x(H_0 < \infty;\ H_x < \infty)\\ &= P_x(H_x = \infty)P_x(H_0 < \infty) - P_x(H_0 < \infty)\\ &\quad + P_x(H_0 < \infty)P_0(H_x < \infty) + P_x(H_x < \infty)P_x(H_0 < \infty)\\ &= F(x,0)F(0,x) = N^{-2d(0,x)}.\end{aligned}$$

Additionlly, by the symmetry of T_N, we have

$$P_x(X_1 \neq x, \cdots, X_{k-1} \neq x,\ X_k = y) = P_y(X_1 \neq y, \cdots, X_{k-1} \neq y,\ X_k = x)$$

for any $x, y \in T_N$, and $k \geq 1$, and moreover

$$P_y(X_1 \neq y, \cdots, X_{k-1} \neq y,\ X_k = x) = P_x(X_1 \neq y, \cdots, X_{k-1} \neq y,\ X_k = y).$$

From these facts, it follows that $p^x_k(x,y) = p^y_k(x,y)$ for all $x, y \in T_N$ and $k \geq 1$. Therefore

$$\begin{aligned}a &= \sum_{j=1}^{\infty}\sum_{x\neq 0} p^x_j(0,x) N^{-2d(0,x)} = \sum_{x\neq 0} F(0,x)N^{-2d(0,x)}\\ &= \sum_{x\neq 0} N^{-3d(0,x)} = (N+1)N^{-3} + \sum_{k=2}^{\infty} N^{-3k}(N+1)N^{k-1} = \frac{1}{N(N-1)}.\end{aligned}$$

Thus, we get $q - q^2 + 2a = (N^2+1)/[N^2(N-1)] = \sigma$. The proof of Proposition 3.1 is completed. ■

§4. Proof of Theorem 1.1.

In this section, we prove both strong law of large numbers and the central limit theorem of R_n.

Proposition 4.1. We have $\lim_{n\to\infty} R_n/n = q$, P_0-a.s.

Proof. From the proof of Proposition 3.1, it follows that $E_0R_n = E_0\zeta_n + E_0\eta_n + 1 = qn + O(1)$, and there is a constant $c \in (0,\infty)$ such that $P_0(|R_n - qn| \geq n^{3/4}) \leq cn^{-1/2}$ for large enough n. By Borel-Cantelli lemma, we have $P_0\big(|n^{-3}R_{n^3} - q| \geq n^{-3/4},\ i.o.\big) = 0$. If $n^3 \leq m \leq (n+1)^3$, then $|R_{n^3} - R_m| \leq (n+1)^3 - n^3 = 3n^2 + 3n + 1$. From this fact, we get $P_0\big(|R_n/n - q| \geq n^{-1/8},\ i.o.\big) = 0$. This yields the desired result. ■

Next, we prove the central limit theorem of R_n. Recall that $\sigma = (N^2+1)/[N^2(N-1)]$ and ζ denotes a standard normal variable.

Proposition 4.2. We have $n^{-1/2}(R_n - E_0R_n) \xrightarrow{(d)} \sigma \cdot \zeta$ as $n \to \infty$.

Let $X(a,b) = \{X_i :\ a \leq i \leq b\}$ for $a, b \in R$, and set $I_n = \#\{X(0,n) \cap X(n,2n)\}$. To prove Proposition 4.2, we need the following lemma.

Lemma 4.3. There is a constant $c_k \in (0,\infty)$ for every $k \geq 1$ such that $E_0I_n^k \leq c_k$ for all $n \geq 1$.

Proof. Without loss of generality, we consider the case $k = 2$ only. It is clear that

$$E_0I_n^2 = \sum_{i=0}^{n} P_0\{X_i \in X(n,2n)\} + 2\sum_{0\leq i<j\leq n} P_0\{X_i,\ X_j \in X(n,2n)\}$$
$$= \sum_{i=0}^{n} P_0\{0 \in X(n-i,2n-i)\} + 2\sum_{0\leq i<j\leq n} P_0\{0,\ X_{j-i} \in X(n-i,2n-i)\}. \tag{4.4}$$

Let $\beta_1 = \inf\{m \geq n-i :\ X_m = 0\}$ and $\beta_2 = \inf\{m \geq n-i :\ X_m = X_{j-i}\}$. Then

$$\begin{aligned} &P_0\{0,\ X_{j-i} \in X(n-i,2n-i)\} \\ &\leq P_0(n-i \leq \beta_1 \leq \beta_2 \leq 2n-i) + P_0(n-i \leq \beta_2 \leq \beta_1 \leq 2n-i) \\ &\leq E_0\big[v_{X_{j-i}} I_{\{n-i\leq\beta_1\leq 2n-i\}}\big] + E_0\big[v_{X_{j-i}} I_{\{n-i\leq\beta_2\leq 2n-i\}}\big], \end{aligned} \tag{4.5}$$

where v_y was defined in the proof of Lemma 3.2. By Proposition 2.1, there is a constant $k_2 \in (0,\infty)$ such that

$$\begin{aligned} &P_0\Big(\max_{0\leq l\leq[(j-i)/2]} d(0,X_l) \leq [(j-i)^{1/2}]\Big) \\ &= P_0\big(\tau_{[(j-i)^{1/2}]} \geq [(j-i)/2]\big) \leq k_2(j-i)^{-3/2}, \quad j > i. \end{aligned}$$

Moreover,

$$\begin{aligned} &P_0(d(0,X_{j-i}) \leq (j-i)^{1/4}) \\ &\leq k_2(j-i)^{-3/2} + P_0\big(\tau_{[(j-i)^{1/2}]} < [(j-i)/2];\ d(0,X_{j-i}) \leq (j-i)^{1/4}\big) \\ &\leq k_2(j-i)^{-3/2} + E_0\big[I_{\{\tau_{[(j-i)^{1/2}]}<[(j-i)/2]\}} P_{X(\tau_{[(j-i)^{1/2}]})}(\beta_3 < \infty)\big], \end{aligned}$$

where $\beta_3 = \inf\{m \geq 0 :\ |X_m| \leq (j-i)^{1/4}\}$. Noticing the structure of T_N and using Lemma 3.2, we get $P_{X(\tau_{[(j-i)^{1/2}]})}(\beta_3 < \infty) \leq k_3(j-i)^{-3/2}$ on $\{\tau_{[(j-i)^{1/2}]} < \infty\}$ for some constant $k_3 \in (0,\infty)$. These two estimates give us $P_0(d(0,X_{j-i}) \leq (j-i)^{1/4}) \leq (k_2+k_3)(j-i)^{-3/2}$. Hence, there is a constant $k_4 \in (0,\infty)$ such that

$$E_0v_{X_{j-i}} = E_0\big[n^{-d(0,X_{j-i})}\big] \leq (k_2+k_3)(j-i)^{-3/2} + N^{-(j-i)^{1/4}} \leq k_4(j-i)^{-3/2}.$$

From the above proof, one also sees that there is a constant $k_5 \in (0,\infty)$ such that

$$\begin{aligned} &P_{X_{j-i}}(n-j \leq \beta_2 \leq 2n-j) \leq k_5(n-j)^{-3/2}, \\ &P_0\{0 \in X(n-i,2n-i)\} \leq k_5(n-i)^{-3/2}. \end{aligned}$$

Therefore, we have

$$\sum_{i=0}^{n} P_0\{0 \in X(n-i, 2n-i)\} \le 1 + k_5 \sum_{i=0}^{n-1} (n-i)^{-3/2} \le c' \tag{4.6}$$

and

$$\begin{aligned}
&E_0\big[v_{X_{j-i}}\, I_{\{n-i\le\beta_2\le 2n-i\}}\big] = E_0\big[v_{X_{j-i}}\, P_{X_{j-i}}(n-j \le \beta_2 \le 2n-j)\big]\\
&\le k_5 (n-j)^{-3/2} E_0(v_{X_{j-i}}) \le k_4 k_5 (j-i)^{-3/2}(n-j)^{-3/2}.
\end{aligned}$$

In virtue of (4.4), (4.5) and (4.6), to complete the proof of the lemma, it is sufficient to prove the following

$$\sum_{0\le i<j\le n} E_0\big[v_{X_{j-i}} I_{\{n-i\le\beta_1\le 2n-i\}}\big] \le c_1', \quad \forall n \ge 1 \tag{4.7}$$

for some constant $c_1' \in (0,\infty)$. Indeed, by Proposition 2.1 we have

$$P_{X_{j-i}}\Big(\max_{0\le l\le[(n-j)/2]} d(X_0, X_l) \le [(n-j)^{1/2}]\Big) \le k_2 (n-j)^{-3/2}.$$

Let $\beta_4 = \inf\{m \ge 0 :\ d(X_0, X_m) \ge [(n-j)^{1/2}]\}$. Then

$$\begin{aligned}
&P_{X_{j-i}}(0 \in X(n-j, 2n-j))\\
&\le k_2 (n-j)^{-3/2} + P_{X_{j-i}}\big(\beta_4 \le [(n-j)/2];\ 0 \in X(n-j, 2n-j)\big)\\
&\le k_2 (n-j)^{-3/2} + E_{X_{j-i}}\big[P_{X(\beta_4)}(H_0 < \infty) I_{\{\beta_4 \le [(n-j)/2]\}}\big].
\end{aligned}$$

If $d(0, X_{j-i}) \le \frac{1}{2}[(n-j)^{1/2}]$, then

$$\begin{aligned}
E_{X_{j-i}}[P_{X(\beta_4)}(H_0<\infty) I_{\{\beta_4<\infty\}}] &\le \max\{P_y(H_0<\infty) :\ y \in T_N,\ d(y, X_{j-i}) = [(n-j)^{1/2}]\}\\
&\le \max\{P_y(H_0<\infty) :\ y \in T_N,\ d(0, y) \ge 1/2[(n-j)^{1/2}]\}\\
&= N^{-\frac{1}{2}[(n-j)^{1/2}]} \qquad \text{(by Lemma 3.2)}.
\end{aligned}$$

Thus, there is a constant $k_6 \in (0,\infty)$ such that $P_{X_{j-i}}(n-j \le \beta_1 \le 2n-j) \le k_6 (n-j)^{-3/2}$, provided $d(0, X_{j-i}) \le \frac{1}{2}[(n-j)^{1/2}]$. Therefore

$$\begin{aligned}
E_0[v_{X_{j-i}} I_{\{n-i\le\beta_1\le 2n-i\}}] &\le N^{-\frac{1}{2}[(n-j)^{1/2}]} P_0(n-i \le \beta_1 \le 2n-i)\\
&\quad + E_0\big[v_{X_{j-i}} I_{\{d(0,X_{j-i})\le\frac{1}{2}[(n-j)^{1/2}]\}} I_{\{n-i\le\beta_1\le 2n-i\}}\big]\\
&\le k_5 (n-i)^{-3/2} N^{-\frac{1}{2}[(n-j)^{1/2}]}\\
&\quad + E_0\big[v_{X_{j-i}} I_{\{d(0,X_{j-i})\le\frac{1}{2}[(n-j)^{1/2}]\}} P_{X_{j-i}}(0 \in X(n-j, 2n-j))\big]\\
&\le k_5 (n-i)^{-3/2} N^{-\frac{1}{2}[(n-j)^{1/2}]} + k_6 (n-j)^{-3/2} E_0(v_{X_{j-i}})\\
&\le k_5 (n-i)^{-3/2} N^{-\frac{1}{2}[(n-j)^{1/2}]} + k_4 k_6 (n-j)^{-3/2} (j-i)^{-3/2}
\end{aligned}$$

which leads to (4.7). We have thus completed the proof of Lemma 4.3. ■

We are now in the position to prove Proposition 4.2. The following argument is based on [6, Proof of Theorem 4.5].

Proof of Proposition 4.2. Given a sufficient small $\delta \in (0,1)$. For each $n \ge 1$, take $p = p(n) = [n^\delta]$. Then, we have

$$R_n = \sum_{i=1}^{p} \#\Big\{X\Big(\frac{i-1}{p}n, \frac{i}{p}n\Big)\Big\} - \sum_{i=2}^{p} \#\Big\{X\Big(0, \frac{i-1}{p}n\Big) \bigcap X\Big(\frac{i-1}{p}n, \frac{i}{p}n\Big)\Big\}.$$

By Lemma 4.3, there is a constant $k_7 \in (0,\infty)$ such that

$$E_0\Big[\Big(\sum_{i=2}^{p}\#\Big\{X\Big(0,\frac{i-1}{p}n\Big)\bigcap X\Big(\frac{i-1}{p}n,\frac{i}{p}n\Big)\Big\}\Big)^2\Big]^{1/2} \le k_7\, p.$$

Thus,

$$E_0\Big|R_n - \sum_{i=1}^{p}\#\Big\{X\Big(\frac{i-1}{p}n,\frac{i}{p}n\Big)\Big\}\Big|^2 = o(n), \quad n\to\infty.$$

Set $R_{n,i} = \#\{X(\frac{i-1}{p}n,\frac{i}{p}n)\},\ 1\le i\le p$, and let $\{R_{n,i}\} = R_{n,i} - E_0R_{n,i}$. From Proposition 3.1, we get $E_0\{R_{n,i}\}^2 \sim \sigma n/p, \quad n\to\infty$.

We now prove that the random variables $R_{n,1},\cdots,R_{n,p}$ are independent. Without loss of generality, we may deal with the independence of $R_{n,1}$ and $R_{n,2}$ only. Indeed, we have $P_0(R_{n,1} = m_1;\ R_{n,2} = m_2) = E_0\big[I_{\{R_{n,1}=m_1\}}P_{X(n/p)}(R_{n,1}=m_2)\big]$. By the symmetry of T_N, one knows that $P_x(R_{n,1}=m_2) = P_0(R_{n,1}=m_2)$ for all $x\in T_N$. Hence

$$\begin{aligned}&P_0(R_{n,1}=m_1;\ R_{n,2}=m_2) = P_0(R_{n,1}=m_1)\cdot P_0(R_{n,1}=m_2)\\ &= P_0(R_{n,1}=m_1)\cdot E_0\big[P_{X(n/p)}(R_{n,1}=m_2)\big]\\ &= P_0(R_{n,1}=m_1)\cdot P_0(R_{n,2}=m_2), \quad \forall m_1,\ m_2\ge 0\end{aligned}$$

which deduces the desired result. Thus, the assertion of Proposition 4.1 holds once the so-called Lindeberg's condition is satisfied for the family $\{R_{n,1}\},\cdots,\{R_{n,p(n)}\}$. Moreover, this condition is satisfied whenever

$$E_0\{R_n\}^4 \le k_8 n^2, \quad \forall n\ge 1 \tag{4.8}$$

for some constant $k_8\in(0,\infty)$. To see this, set $n_1 = [n/2]$. Then

$$(E_0\{R_n\}^4)^{1/4} \le \big(E_0\{\#\{X(0,n_1)\} + \#\{X(n_1,n)\}\}^4\big)^{1/4} + \big(E_0\#\{X(0,n_1)\cap X(n_1,n)\}^4\big)^{1/4}.$$

By Lemma 4.3, we have $\big(E_0\{\#\{X(0,n_1)\cap X(n_1,n)\}\}^4\big)^{1/4} = o(n^{1/2})$. Since $\#\{X(0,n_1)\}$ and $\#\{X(n_1,n)\}$ are independent, from Proposition 3.1, we get

$$E_0\big\{\#\{X(0,n_1)\} + \#\{X(n_1,n)\}\big\}^4 \le E_0\{R_{n_1}\}^4 + E_0\{R_{n_2}\}^4 + k_9 n$$

for some constant $k_9\in(0,\infty)$, where $n_2 = n-n_1$. Thus

$$(E_0\{R_n\}^4)^{1/4} \le \big(E_0\{R_{n_1}\}^4 + E_0\{R_{n_2}\}^4 + k_9 n^2\big)^{1/4} + o(n^{1/2}).$$

For $k\ge 1$, set $\alpha_k = \sup\big\{2^{-k/2}(E_0\{R_n\}^4)^{1/4}:\ 2^k\le n\le 2^{k+1}\big\}$. Then $\alpha_{k+1}\le(1/2\alpha_k^4+C)^{1/4}+O(1)$. This implies that the sequence $\{\alpha_k\}$ is bounded. Therefore, (4.8) holds. We have completed the proof of Proposition 4.2. ■

Proof of Theorem 1.1. Simply combine Proposition 3.1, Proposition 4.1 with Proposition 4.2. ■

§5. Proof of Theorem 1.2.

We begin with several lemmas.

Lemma 5.1. Let $\tilde E_0$ and $\tilde R_n$ be the same as defined in Section 1. We have $\lim_{n\to\infty} n^{-1/2}|E_0R_n - \tilde E_0\tilde R_n| = 0$.

Proof. Take $\epsilon_1\in(0,1/2)$ and $\epsilon_2\in(0,1/2-\epsilon_1)$. By Proposition 2.1, we know that

$$P_0\Big(\max_{0\le l\le[n^{1/2-\epsilon_1}]} d(0,X_l)\le[n^{\epsilon_2}]\Big) \le C(\epsilon_1,\epsilon_2)n^{2\epsilon_1+\epsilon_2}n^{-1},$$

for some constant $C(\epsilon_1,\epsilon_2)\in(0,\infty)$. Let $\nu_n=\inf\{m\geq 0:\ X_m\in S_{[n^{\epsilon_2}]}\}$. By Lemma 3.2, we have $P_{X_{\nu_n}}(H_0<\infty)=N^{-d(0,X_{\nu_n})}=N^{-[n^{\epsilon_2}]}$. By Hölder's inequality, for a fixed $\epsilon_3\in(0,1)$ such that $[1-(2\epsilon_1+2\epsilon_2)](1-\epsilon_3)>1/2$, we have

$$\begin{aligned}
E_0R_n &= E_0\Big[\sum_{i=[n^{1/2-\epsilon_1}]}^{n} I_{\{X_i\neq X_{i+1},\cdots,X_i\neq X_n\}}\Big]+o(n^{1/2})\\
&= E_0\Big[\sum_{i=\nu_n}^{n} I_{\{X_i\neq X_{i+1},\cdots,X_i\neq X_n\}}I_{\{\nu_n\leq[n^{1/2-\epsilon_1}]\}}\Big]+O\big(n^{1-(1-\epsilon_3)}n^{(2\epsilon_1+2\epsilon_2)(1-\epsilon_3)}\big)+o(n^{1/2})\\
&= E_0\bigg(\Big[\sum_{i=\nu_n}^{n} I_{\{X_i\neq X_{i+1},\cdots,X_i\neq X_n\}}\Big]I_{\{X_j\neq 0,j\geq\nu_n\}}I_{\{\nu_n\leq[n^{1/2-\epsilon_1}]\}}\bigg)+o(n^{1/2})\\
&= E_0\bigg[E_{X_{\nu_n}}\Big(\sum_{i=0}^{n} I_{\{X_i\neq X_{i+1},\cdots,X_i\neq X_n\}}I_{\{X_1\neq 0,\cdots,X_n\neq 0\}}\Big)I_{\{\nu_n\leq[n^{1/2-\epsilon_1}]\}}\bigg]+o(n^{1/2}),
\end{aligned}$$

Let $\mu_n=\inf\{m\geq 0:\ Y_m\in S_{[n^{\epsilon_2}]}\}$ and $H_0=\inf\{m\geq 1:\ Y_m=0\}$. Then, it is easy to see that $\tilde{P}_{Y_{\mu_n}}(H_0<\infty)=N^{-[n^{\epsilon_2}]}$. Thus, a similar argument can imply that

$$\tilde{E}_0\tilde{R}_n=\tilde{E}_0\bigg[\tilde{E}_{Y_{\mu_n}}\Big(\sum_{i=0}^{n} I_{\{Y_i\neq Y_{i+1},\cdots,Y_i\neq Y_n\}}I_{\{Y_1\neq 0,\cdots,Y_n\neq 0\}}\Big)I_{\{\mu_n\leq[n^{1/2-\epsilon_1}]\}}\bigg]+o(n^{1/2}).$$

Since T_N and $\tilde{T}_N$ have the same structure except at the point 0, when $\mu_n<\infty$ and $\nu_n<\infty$, we have

$$\begin{aligned}
&E_{X_{\nu_n}}\Big(\sum_{i=0}^{n} I_{\{X_i\neq X_{i+1},\cdots,X_i\neq X_n\}}I_{\{X_1\neq 0,\cdots,X_n\neq 0\}}\Big)\\
&=\tilde{E}_{Y_{\mu_n}}\Big(\sum_{i=0}^{n} I_{\{Y_i\neq Y_{i+1},\cdots,Y_i\neq Y_n\}}I_{\{Y_1\neq 0,\cdots,Y_n\neq 0\}}\Big)
\end{aligned}$$

which leads to $E_0R_n=\tilde{E}_0\tilde{R}_n+o(n^{1/2})$. The proof of Lemma 5.1 is completed. ■

From the construction of T_N, one also sees that there are $T_{1,N},\cdots,T_{N+1,N}$ such that $T_N=\cup_{i=1}^{N+1}T_{i,N}$ and $T_{i,N}\cap T_{j,N}=\{o\}$ for $i\neq j$ and $T_{1,N},\cdots,T_{N,N+1}$ are isomorphic. Let $\{\bar{X}_n\}_{n\geq 0}$ be the simple random walk on $T_{1,N}$ with the probability law $\{\bar{P}_x\}_{x\in T_{1,N}}$, and $r_n=\#\{\bar{X}_0,\bar{X}_1,\cdots,\bar{X}_n\}$.

Lemma 5.2. Let $\sigma=(N^2+1)/[N^2(N-1)]$. Then for every $x\in R^1$, we have

$$\bar{P}_0\Big(\frac{r_n-E_0R_n}{\sigma n^{1/2}}\leq x\Big)\longrightarrow(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)dy,\quad n\to\infty.$$

Proof. By Proposition 4.2

$$P_0\Big(\frac{R_n-E_0R_n}{\sigma n^{1/2}}\leq x\Big)\longrightarrow(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)dy,\quad n\to\infty,\quad\forall x\in R^1.$$

Set $g_n=\#\{X_{\nu_n},\cdots,X_{\nu_n+n}\}I_{\{X_i\neq 0,\ \forall i\geq\nu_n;\ \nu_n\leq[n^{1/2-\epsilon_1}]\}}$. From the proof of Lemma 5.1, one sees that $\sigma^{-1}n^{-1/2}E_0|g_n-R_n|=o(1)$ as $n\to\infty$. Hence $P_0\big(\sigma^{-1}n^{-1/2}(g_n-E_0R_n)\leq x\big)\to (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)dy,\ n\to\infty,\ x\in R^1$. Let $t_n=\inf\{m\geq 0:\ d(0,\bar{X}_m)=n\}$ and set $f_n=\#\{\bar{X}_{t_n},\cdots,\bar{X}_{t_n+n}\}I_{\{X_i\neq 0,\ \forall i\geq t_n;\ t_n\leq[n^{1/2-\epsilon_1}]\}}$. Then the random variables f_n g_n have the same distribution, Therefore

$$\bar{P}_0\big(\sigma^{-1}n^{-1/2}(f_n-E_0R_n)\leq x\big)\longrightarrow(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)dy,\quad n\to\infty,\quad\forall x\in R^1.$$

By a similar argument as in the proof of Lemma 5.1, we can prove $\bar{E}_0|f_n - r_n| = o(n^{1/2})$, $n \to \infty$, where $\bar{E}_x$ is the expectation with respect to $\bar{P}_x$. In other words, we have $n^{-1/2}(f_n - r_n) \xrightarrow{\bar{P}_0} 0$, $n \to \infty$. This yields the desired result. ∎

Having these preparations, we can complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Firstly, we prove

$$\tilde{P}_0\Big(\frac{\tilde{R}_n - \tilde{E}_0\tilde{R}_n}{\sigma n^{1/2}} \le x\Big) \to (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)dy, \quad n \to \infty,\ \forall x \in R^1. \tag{5.3}$$

Let $h_n = \#\{Y_{\mu_n}, \cdots, Y_{\mu_n+n}\}I_{\{Y_i \ne 0,\ \forall i \ge \mu_n;\ \mu_n \le [n^{1/2-\epsilon_1}]\}}$. Then, we have $\tilde{E}_0|\tilde{R}_n - h_n| = o(n^{1/2})$, $n \to \infty$, which implies $n^{-1/2}(\tilde{R}_n - h_n) \xrightarrow{\tilde{P}_0} 0$, $n \to \infty$. In addition, h_n and f_n have the same distribution too. Thus, Lemma 5.2 implies that

$$\tilde{P}_0\Big(\frac{h_n - E_0R_n}{\sigma n^{1/2}} \le x\Big) \to (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)dy, \quad n \to \infty, \quad \forall x \in R^1.$$

Hence, (5.3) follows from Lemma 5.1 immediately.

Next, we prove

$$\tilde{E}_0(\tilde{R}_n - \tilde{E}_0R_n)^2/n \longrightarrow \sigma, \quad n \to \infty. \tag{5.4}$$

In fact, from the proof of Lemma 5.1 one sees that $E_0|R_n - g_n|^2 = o(n)$, $n \to \infty$. Thus, Proposition 4.2 yields $n^{-1}E_0|g_n - E_0R_n|^2 \to \sigma$, $n \to \infty$, which implies that $n^{-1}\bar{E}_0|f_n - E_0R_n|^2 \to \sigma$, $n \to \infty$. Therefore, $n^{-1}\tilde{E}_0|h_n - E_0R_n|^2 \to \sigma$, $n \to \infty$. From the proof of Lemma 5.1, one can also see that $n^{-1}\tilde{E}_0|\tilde{R}_n - h_n|^2 \to 0$, $n \to \infty$. Thus, we get (5.4) immediately from Lemma 5.1. We have completed the proof of Theorem 1.2. ∎

Remark 5.5. Let $T'_N = (E', V'), T_N = (E, V)$, and $\{X'_n\}_{n\ge 0}$ be the simple random walk on T'_N. Suppose that $\#\{(E' \setminus E) \cup (E \setminus E')\} < \infty$ and $\#\{(V' \setminus V) \cup (V \setminus V')\} < \infty$. Then, from the above arguments we see that the conclusions of Theorem 1.1 hold if R_n is replaced by $R'_n = \#\{X'_0, \cdots, X'_n\}$.

§6. Trapping problem. Proof of Theorem 1.3.

In the present section, we study the trapping problem on trees. The main aim is to complete the proof of Theorem 1.3. As stated in [4], an accurate approximation to the survival probability at short times is quite valuable for physical applications. Due to this reason, in this paper we only concern with the asymptotic behaviour of survival probability for moderately large n and small ϵ and that of expected trapping time for small ϵ.

It is easy to check that $f(n) = E_0(1-\epsilon)^{R_n} = E_0\big[\exp(R_n\log(1-\epsilon))\big] = E_0[\exp(-\lambda R_n)]$, where $\lambda = \log(1-\epsilon)^{-1}$. As in [4], we can write $f(n) = \sum_{j=0}^{\infty}(-1)^j\lambda^jE_0R_n^j/j! =: \exp[K(\lambda, n)]$, where $K(\lambda, n) = \sum_{j=1}^{\infty}(-1)^j\lambda^jk_j(n)/j!$ and $k_j(n)$ can be defined in terms of E_0R_n and the centering moments $E_0(R_n - E_0R_n)^i$ of order $i \le j$. As a fact, one can check that

$$k_1(n) = E_0R_n, \quad k_j(n) = E_0(R_n - E_0R_n)^j, \quad j = 2, 3,$$
$$k_4(n) = E_0(R_n - E_0R_n)^4 - 3[E_0(R_n - E_0R_n)^2]^2.$$

Recall that (Proposition 4.1) $\lim_{n\to\infty} n^{-1}E_0R_n = (N-1)/N$ and the fact: $\lim_{\epsilon\to 0^+}\epsilon^{-1}\lambda = 1$. For small ϵ and moderately large n $\log f(n) \sim -\epsilon n(N-1)/N$, which proves the first part of Theorem 1.3.

Next, we consider the expected trapping time. By definition, we have $ET = \sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} E_0[\exp(-\lambda R_n)]$. By Jensen's inequality, we have $ET \ge \sum_{n=0}^{\infty}\exp(-\lambda E_0R_n)$. From the proof of Proposition 4.1, we know $E_0R_n = n(N-1)/N+O(1)$, $n\to\infty$. Thus, there is a constant $K\in(0,\infty)$ such that $E_0R_n \le n(N-1)/N+K$ for all $n\ge 0$. Hence

$$\begin{aligned}\varliminf_{\epsilon\to 0^+}\epsilon ET &\ge \varliminf_{\epsilon\to 0^+}\Big[\epsilon\exp(-\lambda K)\sum_{n=0}^{\infty}\exp\Big(-\lambda\frac{N-1}{N}n\Big)\Big]\\ &= \varliminf_{\epsilon\to 0^+}\Big[\epsilon\exp(-\lambda K)\Big(1-\exp\Big(-\lambda\frac{N-1}{N}\Big)\Big)^{-1}\Big] = \frac{N}{N-1}.\end{aligned}$$

As in [3], we let $I_1(\epsilon,n)=\sum_{0\le n\le M\epsilon^{-1}}f(n)$ and $I_2(\epsilon,n)=\sum_{n>M\epsilon^{-1}}f(n)$. Then, $ET=I_1(\epsilon,n)+I_2(\epsilon,n)$. Clearly, the following desired result $\overline{\lim}_{\epsilon\to0^+}\epsilon ET\le N/(N-1)$ can be deduced from

$$\lim_{M\to\infty}\varlimsup_{\epsilon\to0^+}\epsilon I_2(\epsilon,n)=0 \tag{6.1}$$

$$\varlimsup_{M\to\infty}\varlimsup_{\epsilon\to\infty}\epsilon I_1(\epsilon,n)\le N/(N-1). \tag{6.2}$$

To prove (6.1) and (6.2), we need a lemma.

Lemma 6.3. There is a constant $K_1>0$ such that $P_0(R_n\le h(n))\le\exp(-K_1n/h(n))$ for all $n\ge2$ and for any $h(n)$ with $\lim_{n\to\infty}h(n)=\infty$ and $h(n)=o(n)$ as $n\to\infty$.

Proof. From Proposition 3.1 and Proposition 4.1, it follows that large enough n, $P_0\big(R_n\le(N-1)n/(4N)\big)\le 1/2$. Let $R^j(n)=\#\big\{X_m:\ j\delta n\le m\le(j+1)\delta n\big\}$. From the proof of Proposition 4.2, we see that $R^0(n),R^1(n),\cdots,R^{[1/\delta]}(n)$ are independent. Take $\delta=4\frac{N}{N-1}h(n)n^{-1}$. Then, for large enough n,

$$\begin{aligned}P_0(R_n\le h(n)) &\le P_0\Big(\max_{0\le j\le\frac14\frac{N-1}{N}h(n)n^{-1}}R^j\Big(\frac{4Nh(n)}{N-1}\Big)\le h(n)\Big)\\ &=\Big[P_0\Big(R^0\Big(\frac{4Nh(n)}{N-1}\Big)\le h(n)\Big)\Big]^{\frac{(N-1)n}{4h(n)N}}\le 2^{-\frac{(N-1)n}{4Nh(n)}}\end{aligned}$$

which proves the desired result. ■

Proof of Theorem 1.3. Setting $h(n)=(K_1n/\epsilon)^{1/2}$ in Lemma 6.3, it follows that for large enough n, $f(n)\le E_0(\exp(-\lambda R_n))\le 2\exp\big(-(K_1\epsilon n/2)^{1/2}\big)$. Hence

$$\begin{aligned}I_2(\epsilon,n) &\le 2\sum_{n>M\epsilon^{-1}}^{\infty}\exp\big(-(K_1\epsilon n/2)^{1/2}\big)\le 2\int_{M\epsilon^{-1}/2}^{\infty}\exp\big(-(K_1\epsilon x/2)^{1/2}\big)dx\\ &=2\epsilon^{-1}\int_{M/2}^{\infty}\exp\big(-(K_1x/2)^{1/2}\big)dx,\end{aligned}$$

which implies (6.1).

It is clear that for sufficient small ϵ and $0\le n\le\epsilon^{-1}M$, $\epsilon E_0R_n\le\epsilon(N-1)n/N+\epsilon K\le 2M(N-1)/N$. Moreover, there is a constant $K_2\in(0,\infty)$ such that $|\epsilon R_n-\epsilon E_0R_n|\le\epsilon K_2E_0R_n$ for all $n\ge1$ and $\epsilon>0$. By Proposition 3.1, we can show $E_0|R_n-E_0R_n|^k\le|K_2E_0R_n|^{k-2}var(R_n)\le K_3n^{-1}(K_2E_0R_n)^k$ for all $k\ge2$ and some constant $K_3\in(0,\infty)$. Moreover, one has $E_0|R_n-E_0R_n|\le(E_0|R_n-E_0R_n|^2)^{1/2}\le K_4n^{-1/2}(K_2E_0R_n)$ for all $n\ge1$ and some constant $K_4\in(0,\infty)$. Therefore, if $M^{-1}\epsilon^{-1}\le n\le M\epsilon^{-1}$, then

$$\begin{aligned}&E_0\exp(-\lambda(R_n-E_0R_n))=\textstyle\sum_{k=0}^{\infty}(-1)^k\lambda^kE_0(R_n-E_0R_n)^k/k!\\ &\le 1+\max(K_3,K_4)n^{-1/2}\exp(\lambda K_2E_0R_n)\\ &\le 1+\max(K_3,K_4)n^{-1/2}\exp\big(2(N-1)K_2M/N\big).\end{aligned}$$

By Cauchy inequality, we have

$$\begin{aligned} E_0 \exp(-\lambda R_n) &\le \left[E_0 \exp(-2\lambda(R_n - E_0 R_n))\right]^{1/2} \exp(-\lambda E_0 R_n) \\ &\le \left[1 + \max(K_3, K_4) n^{-1/2} \exp(4(N-1)K_2 M/N)\right]^{1/2} \exp(-\lambda E_0 R_n), \end{aligned}$$

provided $M^{-1}\epsilon^{-1} \le n \le M\epsilon^{-1}$. Thus

$$\begin{aligned} I_1(\epsilon, n) - M^{-1}\epsilon^{-1} \ &\le \sum_{M^{-1}\epsilon^{-1} \le n \le M\epsilon^{-1}} \exp(-\lambda E_0 R_n) \\ &\cdot \left[1 + \max(K_3, K_4)\epsilon^{1/2} M^{1/2} \exp\big(4(N-1)K_2 M/N\big)\right]^{1/2} \\ &\sim \epsilon^{-1}\left[1 + \epsilon^{1/2}\max(K_3, K_4) M^{1/2} \exp\big(4(N-1)K_2 M/N\big)\right]^{1/2} \int_{\frac{1}{2}M^{-1}}^{2M} \exp\Big(-\frac{N-1}{N}x\Big)dx, \end{aligned}$$

which implies the desired result (6.2). The proof of Theorem 1.3 is completed. ■

In a similar way, one may discuss the trapping problem on $\tilde{T}_N$. Suppose that the random trap field $(C(x))_{x\in\tilde{T}_N}$ with density $\epsilon > 0$ is on the tree $\tilde{T}_N$. Let $\tilde{T} = \inf\{n \ge 0 : \ C(Y_n) = 1\}$ and $\tilde{f}(n) = \tilde{P}(\tilde{T} > n)$, where $\tilde{P} = \tilde{P}_0 \times P_C$. Then we have

Corollary 6.4. i) For small $\epsilon > 0$ and moderately large n, we have $\log \tilde{f}(n) \sim \epsilon n(N-1)/N$.
ii) Let $\tilde{E}$ is the expectation with respect to $\tilde{P}$, we have $\lim_{\epsilon\to 0^+} \epsilon \cdot \tilde{E}\tilde{T} = N/(N-1)$.

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