CONTINUUM LIMIT FOR REACTION DIFFUSION PROCESSES WITH SEVERAL SPECIES

Chen Mu Fa
(Beijing Normal University)

Huang Li Ping
(Hubei University)

Xu Xian Jin
(Beijing Normal University)

Abstract

The continuum limit for reaction diffusion processes with several species is studied. It is also proved that the propagation of chaos holds at any time.

1. Introduction

In recent years, the reaction diffusion processes have been studied by many authors (see [2], [3] and the references within). More recently, Han Dong[4] has constructed the Markov processes for a quite general class of reaction diffusion processes with several species, meanwhile, C.Boldrighini et al [1] and some others studied the continuum limit for the reaction diffusion processes with only one species. As C.Boldrighini et al pointed out in [1] that the problem of the continuum limit for the reaction diffusion processes with several species is still open to discussion. This motivated us to write this paper. In this paper some related problems under appropriate space-time (continuum) scaling are also considered.

Here, we need some notations to introduce our main result.

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_+$ be the set of non-negative ones. Let $E = (\mathbb{Z}_+^2)^{\mathbb{Z}}$ and $\Omega^\varepsilon = \varepsilon^{-2}\Omega_d + \Omega_r$ $(\varepsilon > 0)$ be the formal generator of the processes considered in the paper, its diffusion part Ω_d and reaction part Ω_r are given as follows:

$$\Omega_d f(\eta) = \sum_{i=1,2} \sum_{x\in\mathbb{Z}} \eta_i(x) \sum_{y\in\mathbb{Z}} p(x,y)[f(\eta - e_x^i + e_y^i) - f(\eta)],$$

$$\Omega_r f(\eta) = \sum_{i,j=1,2} \sum_{x\in\mathbb{Z}} [a_i \binom{\eta_i(x)}{2} + b_i \eta_1(x)\eta_2(x)]\cdot [f(\eta - e_x^i + e_x^j) - f(\eta)], \eta \in E. \tag{1.1}$$

where $a_i, b_i \geq 0$, $p(x,y)$ is the transition probability of the simple random walk on $\mathbb{Z}$. $e_x^1 = (e_x, \theta)$, $e_x^2 = (\theta, e_x)$, e_x is the unit vector in $\mathbb{Z}_+^{\mathbb{Z}}$ with value 1 at x and θ is the zero vector in $\mathbb{Z}_+^{\mathbb{Z}}$.

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Let

$$E_0 = \{\eta \in E : \|\eta\| = \sum_{x\in\mathbb{Z}} (\eta_1(x) + \eta_2(x))2^{-|x|} < \infty\}.$$

Denote by P^ε_η the probability of the Markov process with generator Ω^ε and state space E_0 starting from $\eta \in E_0$ (see [2] and [4]).

Following [1], we will introduce some polynomials on $\mathbb{Z}_+^{\mathbb{Z}} \times \mathbb{Z}^k (k \in \mathbb{Z}_+)$. For $\eta \in \mathbb{Z}_+^{\mathbb{Z}}$ and $x = (z, \cdots, z) \in \mathbb{Z}^k$, set

$$p(\eta, x, k) = \eta(z)(\eta(z) - 1)\cdots(\eta(z) - k + 1).$$

For $x = (x_1, \cdots, x_k) \in \mathbb{Z}^k$ which consists of different elements, set

$$p(\eta, x, k) = \prod_{i=1}^{m} p(\eta, x_{l_i}, k_i), \tag{1.2}$$

where $\{x_{l_i}\}_{i=1}^m$ are the set of distinct elements of $\{x_l : 1 \le l \le k\}$ with multiplicity k_i.

Throughout this paper, we restrict ourselves to the initial measure $\mu^\varepsilon(\varepsilon > 0)$ satisfying the following two hypotheses:

(H_1). Initial asymptotic condition: For any $L > 0, k, l \in \mathbb{Z}_+$,

$$\lim \max_{(L,\varepsilon)} |\mu^\varepsilon(p(\eta_1, x, k)p(\eta_2, y, l)) - \prod_{i=1}^{k} \rho_1(\varepsilon x_i) \prod_{i=1}^{L} \rho_2(\varepsilon y_i)| = 0 \tag{1.3}$$

where $\max_{(L,\varepsilon)}$ denotes the maximum over $x \in \mathbb{Z}^k$, $y \in \mathbb{Z}^l$ such that $\|\varepsilon x\|, \|\varepsilon y\| \le L$ and $(x, y) \in \mathbb{Z}^{k+l}$ are mutually distinct, $\{\rho_i\}$ are assumed to be uniformly bounded and belong to $C^2(\mathbb{R})$.

(H_2). Initial moment condition: There exists a $C_0 > 0$ such that for any $\varepsilon > 0, k \in \mathbb{Z}_+ x \in \mathbb{Z}^k$,

$$\mu^\varepsilon(p(\eta_1 + \eta_2, x, k)) \le C_0^k \tag{1.4}$$

Theorem: Let $E^\varepsilon_{\mu^\varepsilon}$ be the expectation with respect to $\{P^\varepsilon_\eta, \eta \in E_0\}$ and initial measure $\mu^\varepsilon(\varepsilon > 0)$. Under the hypotheses (H_1) and (H_2), for any $r \in \mathcal{R}, t > 0, i = 1, 2$, we have

$$\lim_{\varepsilon\to 0} E^\varepsilon_{\mu^\varepsilon} \eta_t^{(i)}([\varepsilon^{-1} r]) = f_i(r, t), i = 1, 2. \tag{1.5}$$

where $(f_1(r,t), f_2(r,t))$ solves the following reaction diffusion equations:

$$\begin{aligned}
\frac{\partial f_1(r,t)}{\partial t} =& \frac{1}{2}\Delta f_1(r,t) - \frac{1}{2}a_1 f_1^2(r,t) + \frac{1}{2}a_2 f_2^2(r,t) \\
&+ (b_2 - b_1)f_1(r,t)f_2(r,t) \\
\frac{\partial f_2(r,t)}{\partial t} =& \frac{1}{2}\Delta f_2(r,t) + \frac{1}{2}a_1 f_1^2(r,t) - \frac{1}{2}a_2 f_2^2(r,t) \\
&+ (b_1 - b_2)f_1(r,t)f_2(r,t) \\
f_i(r,0) =& \rho_i(r), \quad i = 1, 2.
\end{aligned} \tag{1.6}$$

Furthermore, for any $k, l \in \mathbb{Z}_+, r \in \mathbb{R}^k, q \in \mathbb{R}^l$,

$$\begin{aligned} &\lim_{\varepsilon\to 0} E^{\varepsilon}_{\mu^{\varepsilon}}[p(\eta_t^{(1)}, [\varepsilon^{-1}r], k)p(\eta_t^{(2)}, [\varepsilon^{-1}q], l)] \\ &= \prod_{i=1}^{k} f_1(r_i, t) \prod_{i=1}^{l} f_2(q_i, t), \end{aligned} \tag{1.7}$$

where $[r] = ([r_i]) \in \mathbb{Z}^k$, $[r_i]$ is the integer part of r_i. The conclusion (1.7) shows that the propagation of chaos holds at any time t.

(1.1) Remark: As an example for the initial measure μ^{ε} considered above, let $\mu_i^{\varepsilon}(x)$ be the Poisson measure with density $\rho_i(\varepsilon x), i = 1, 2$, then $\mu^{\varepsilon} = \prod_{x\in\mathbb{Z}}(\mu_1^{\varepsilon}(x) \cdot \mu_2^{\varepsilon}(x))$ satisfies the hypotheses (H_1) and (H_2).

(1.2) Remark: It should be pointed out that the quadric degree in the formulation of the reaction part Ω_r is not essential. The linear case is much easier. The reason for choose the quadric case is to show the main idea for the non- linear cases. Certainly, more general models can be handled in a similar way.

The paper is organized as follows:

We introduce a duality of the diffusion part Ω_d and independent random walks for function $p(\eta_1 + \eta_2, x, k)$ and $p(\eta_1, x, k) \cdot p(\eta_2, y, l)$, and at the same time we present an integration by parts formula. Then, we prove that $E^{\varepsilon}_{\mu^{\varepsilon}}[p(\eta_t^{(1)}, [\varepsilon^{-1}r], k)p(\eta_t^{(2)}, [\varepsilon^{-1}q], l)]$ are uniformly bounded and equicontinuous on any compact set of $\mathbb{R}^{k+1}$. This gives us the tightness. Based ourselves on these facts, we prove in section 3 that the limits satisfy equations (1.6) and (1.7).

2. Some Basic Lemmas:

Let us consider the diffusion part Ω_d. Set $\bar{\eta} = \eta_1 + \eta_2$, then it is easy to check that for $f_{x,y}(\eta) = p(\eta_1, x, k)p(\eta_2, y, l)$ and $\bar{f}_x(\eta) = p(\bar{\eta}, x, k)$, we have

$$\begin{aligned} \Omega_d f_{x,y}(\eta) =& [\frac{1}{2}\sum_{i=1}^{k} \Delta_i p(\eta_1, x, k)]p(\eta_2, y, l) \\ &+ p(\eta_1, x, k)[\frac{1}{2}\sum_{i=1}^{l} \Delta_i p(\eta_2, y, l)] \end{aligned} \tag{2.1}$$

and

$$\Omega_d \bar{f}_x(\eta) = \frac{1}{2}\sum_{i=1}^{k} \Delta_i p(\bar{\eta}, x, k) \tag{2.2}$$

where Δ_i is the discrete Laplacian acting on the variable x_i. The formula (2.1) has appeared in [1] for single species. By (2.1) and (2.2), we obtain the following formulas:

$$\begin{aligned} &E^{d}_{\mu^{\varepsilon}} p(\eta_t^{(1)}, x, k)p(\eta_t^{(2)}, y, l) \\ &= \sum_{z\in\mathbb{Z}^k, w\in\mathbb{Z}^l} \pi(t, (x, y), (z, w))\mu^{\varepsilon}(p(\eta_1, z, k)p(\eta_2, w, l)), \end{aligned} \tag{2.3}$$

$$E^d_{\mu^\epsilon} p(\bar\eta_t, x, k) = \sum_{y\in \mathbb{Z}^k} \pi(t,x,y)\mu^\epsilon(p(\bar\eta, y, k)), \tag{2.4}$$

where $E^d_{\mu^\epsilon}$ is the expectation with respect to $P^d_{\mu^\epsilon}$ which is the probability measure of the Markov process with generator Ω_d and initial measure μ^ϵ, and for any k and $x = (x_1, \cdots, x_k) \in \mathbb{Z}^k$, $\pi(t,x,\cdot) = \prod_{i=1}^k \pi(t, x_i, \cdot)$, $\pi(t,x_i,\cdot)$ is the transition probability of the Markov chain with Q-matrix $Q = P - I$.

Let Λ be a finite subset of $\mathbb{Z}$, denoted by $\Lambda \subset\subset \mathbb{Z}$. Consider the finite dimensional reaction diffusion process $P^{\Lambda,\epsilon}_\eta$ with generator $\epsilon^{-2}\Omega_d + \Omega_r$ on $(\mathbb{Z}^2_+)^\Lambda$. Of course, (2.3) and (2.4) hold for the Q- process $P^{\Lambda,d}_\eta$ with generator Ω_d on $(\mathbb{Z}^2_+)^\Lambda$. So we get the integration by parts formulas (see [2]) as follows:

$$\begin{aligned} &E^{\Lambda,\epsilon}_{\mu^\epsilon} p(\eta^{(1)}_t, x, k) p(\eta^{(2)}_t, y, l) \\ &= \sum_{z\in\Lambda^k, w\in\Lambda^l} \pi^\Lambda(\epsilon^{-2}t, (x,y),(z,w))\mu^\epsilon(p(\eta_1, z, k)p(\eta_2, w, l)) \\ &+ \int_0^t ds[\sum_{z\in\Lambda^k, w\in\Lambda^l} \pi^\Lambda(\epsilon^{-2}s,(x,y),(z,w))E^{\Lambda,\epsilon}_{\mu^\epsilon}\Omega_r f_{z,w}(\eta_{t-s})], \end{aligned} \tag{2.5}$$

$$\begin{aligned} &E^{\Lambda,\epsilon}_{\mu^\epsilon} p(\bar\eta_t, x, k) \\ &= \sum_{y\in\Lambda^k} \pi^\Lambda(\epsilon^{-2}t, x, y)\mu^\epsilon(p(\bar\eta, y, k)) \\ &+ \int_0^t ds[\sum_{y\in\Lambda^k} \pi^\Lambda(\epsilon^{-2}s, x, y) E^{\Lambda,\epsilon}_{\mu^\epsilon}\Omega_r \bar f_y(\eta_{t-s})] \end{aligned} \tag{2.6}$$

where $E^{\Lambda,\epsilon}_{\mu^\epsilon}$ is the expectation with respect to $P^{\Lambda,\epsilon}_\eta$ with initial measure μ^ϵ, and for $x \in \Lambda^k$, $\pi^\Lambda(t,x,\cdot) = \prod_{i=1}^k \pi^\Lambda(t,x_i,\cdot)$, $\pi^\Lambda(t,x_i,\cdot)$ is the transition probability of the Markov chain with Q-matrix Q^Λ, the restriction of $Q = P - I$, $f_{x,y}$ and $\bar f_x$ have been given at the beginning of this section.

(2.1). Remark: It is known that $\pi^\Lambda(t,x,\cdot)$ converges to $\pi(t,x,\cdot)$ and the finite dimensional distributions of $P^{\Lambda,\epsilon}_\eta$ converge to these of P^ϵ_η weakly as Λ tends to $\mathbb{Z}$ (see [2] and [4]). Letting $\Lambda \uparrow \mathbb{Z}$ in (2.5) and (2.6), by the moment estimates of $P^{\Lambda,\epsilon}_\eta$ and P^ϵ_η (see [2]), we get (2.5) and (2.6) for the reaction diffusion process $\{P^\epsilon_\eta : \eta \in E_0\}$ with initial measure μ^ϵ.

Lemma 1: Under the hypothesis (H_2), for any $k, l \in \mathbb{Z}_+$, $x \in \mathbb{Z}^k$, $y \in \mathbb{Z}^l$ and $\epsilon > 0$, we have

$$\begin{aligned} &E^\epsilon_{\mu^\epsilon} p(\eta^{(1)}_t, x, k) p(\eta^{(2)}_t, y, l) \\ &\le E^\epsilon_{\mu^\epsilon} p(\bar\eta_t, (x,y), k+l) \\ &\le C_0^{k+l} \end{aligned} \tag{2.7}$$

Proof: For the first inequality in (2.7), it needs only to check that for $x_i = y_j (i = 1, \cdots, k, j = 1, \cdots, l)$ and $\eta = (\eta_1, \eta_2) \in E_0$,

$$\begin{aligned} &p(\eta_1, x, k) p(\eta_2, y, l) \\ &\le p(\eta_1, x, k)(\bar\eta(x_1) - k) \cdots (\bar\eta(x_1) - k - l + 1) \\ &\le p(\bar\eta, (x,y), k+l) \end{aligned}$$

By the integration by parts formula,

$$\begin{aligned}&E^{\varepsilon}_{\mu^{\varepsilon}} p(\overline{\eta}_t, x, k+l)\\ =&\sum_{y\in\mathbb{Z}^{k+l}} \pi(\varepsilon^{-2}t, x, y)\mu^{\varepsilon}(p(\bar{\eta}, y, k+l))\\ \leq& C_0^{k+l}\end{aligned}$$

(2.2). Remark: If each species contains birth and death rates which are independent of other species and are also polynomials, then the upper bound estimate in inequality (2.7) be replaced by $(C_0 e^{ct})^k + O(\varepsilon)$ (see [1] and [2]).

Lemma 2: Under (H_1) and (H_2), we have for any $L > 0, T > 0, k, l \in \mathbb{Z}_+$,

$$\begin{aligned}\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\sup_{(L,T,\delta)} &|E^{\varepsilon}_{\mu^{\varepsilon}}[p(\eta_t^{(1)}, x^{(1)}, k)p(\eta_t^{(2)}, x^{(2)}, l)]\\ &- E^{\varepsilon}_{\mu^{\varepsilon}}[p(\eta_{t'}^{(1)}, y^{(1)}, k)p(\eta_{t'}^{(2)}, y^{(2)}, l)]| = 0\end{aligned} \tag{2.8}$$

where $\sup_{(L,T,\delta)}$ denotes the supremum over all $x^{(1)}, y^{(1)} \in \mathbb{Z}^k$, $x^{(2)}, y^{(2)} \in \mathbb{Z}^l$ and $t, t' \leq T$ such that $\|x^{(i)}\|$, $\|y^{(i)}\| \leq \varepsilon^{-1}L$, $\|x^{(i)} - y^{(i)}\| \leq \varepsilon^{-1}\delta$ and $|t - t'| \leq \delta$.

Proof: We prove (2.8) only in the case that $k = 1$ and $l = 0$. It is similar for the other k and l's. We first prove (2.8) for $t = t'$.

Given $\varepsilon > 0, \delta > 0, |x_1|, |x_2| \leq \varepsilon^{-1}L$, $t \leq T$ and $|x_1 - x_2| \leq \varepsilon^{-1}\delta$, by the integration by parts formula, we have

$$\begin{aligned}&|E^{\varepsilon}_{\mu^{\varepsilon}}\eta_t^{(1)}(x_1) - E^{\varepsilon}_{\mu^{\varepsilon}}\eta_t^{(1)}(x_2)|\\ \leq& |\sum_y [\pi(\varepsilon^{-2}t, x_1, y) - \pi(\varepsilon^{-2}t, x_2, y)]\mu^2(\eta_1(y))|\\ &+ \int_0^t ds \sum_y |\pi(\varepsilon^{-2}s, x_1, y) - \pi(\varepsilon^{-2}s, x_2, y)||E^{\varepsilon}_{\mu^{\varepsilon}}\Omega_r f_y^{(1)}(\eta_{t-s})|\\ =& I_{\varepsilon}(x_1, x_2, t) + II_{\varepsilon}(x_1, x_2, t)\end{aligned}$$

where $f_x^{(1)}(\eta) = \eta_1(x)$. Since

$$\begin{aligned}I_{\varepsilon}(x_1, x_2, t) \leq& |\sum_y \pi(\varepsilon^{-2}t, x_2, y)[\rho_1(\varepsilon(y + x_1 - x_2)) - \rho_1(\varepsilon y)]|\\ &+ \sum_y \pi(\varepsilon^{-2}t, 0, y)[\sum_{i=1,2} |\mu^{\varepsilon}(\eta_1(y + x_i)) - \rho_1(\varepsilon(y + x_i))|]\end{aligned}$$

by (H_1) and (H_2), it follows that

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\sup_{(L,T,\delta)} I_{\varepsilon}(x_1, x_2, t) = 0.$$

On the other hand, for any $t > 0, x_1, x_2 \in \mathbb{Z}$, we have

$$\sum_y |\pi(t, x_1, y) - \pi(t, x_2, y)| \leq C|x_1 - x_2|/\sqrt{t},$$

where C is a constant. By the moment estimate (2.7), we still have

$$\lim_{\delta\geq 0}\overline{\lim_{\delta\to 0}}\sup_{(L,T,\delta)} II_\varepsilon(x_1,x_2,t)=0.$$

Now, suppose that $t<t'$, by considering t as the initial time and using the integration by parts formula and the above result, we can get (2.8) easily #

(2.3). Remark: The proof of (2.8) is similar to those given in [1] for the case of single species. From (2.7) and (2.8), we see that for any $k,l\in\mathbb{Z}_+$, $E^\varepsilon_{\mu^\varepsilon}p(\eta^{(1)}_t,[\varepsilon^{-1}r],k)p(\eta^{(2)}_t,[\varepsilon^{-1}q],l)$ is uniformly bounded and equicontinuous in any bounded set of $\mathbb{R}^{k+l}\times[0,\infty)$.

3. Proof of the Theorem

(I). First of all, we have the following integration by parts formula which is very similar to the integral reaction diffusion equations (1.6):

$$\begin{aligned}
&E^\varepsilon_{\mu^\varepsilon}p(\eta^{(1)}_t,[\varepsilon^{-1}r],k)p(\eta^{(2)}_t,[\varepsilon^{-1}q],l)\\
&=\sum_{x\in\mathbb{Z}^k,y\in\mathbb{Z}^l}\pi(\varepsilon^{-2}t,([\varepsilon^{-1}r],[\varepsilon^{-1}q]),(x,y))\mu^\varepsilon(p(\eta_1,x,k)p(\eta_2,y,l))\\
&\quad+\int_0^t ds\sum_{x\in\mathbb{Z}^k,y\in\mathbb{Z}^l}\pi(\varepsilon^{-2}s,([\varepsilon^{-1}r],[\varepsilon^{-1}q]),(x,y))E^\varepsilon_{\mu^\varepsilon}\Omega_r f_{x,y}(\eta_{t-s})
\end{aligned}\tag{3.1}$$

where $f_{x,y}(\eta)=p(\eta_1,x,k)p(\eta_2,y,l)$.

Denote by $\sum'$ the sum over the set $\{z\in\mathbb{Z}^k: z_i=z_j \text{ for some } i\neq j\}$. It is not difficult to see that for any $k>0$ and $x\in\mathbb{Z}^k$, there exists a $C_k>0$ such that

$$\sum_y{}'\pi(t,x,y)\leq C_k/\sqrt{t}$$

Therefore, by the estimate (2.7) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{(x,y)}{}'\pi(\varepsilon^{-2}s,([\varepsilon^{-1}r],[\varepsilon^{-1}q]),(x,y))\\
&\qquad\cdot[E^\varepsilon_{\mu^\varepsilon}\Omega_r f_{x,y}(\eta_{t-s})]=0
\end{aligned}\tag{3.2}$$

On the other hand, for any $x\in\mathbb{Z}^k$ and $y\in\mathbb{Z}^l$ having completely distinct elements, we have

$$\begin{aligned}
&\Omega_r f_{x,y}(\eta)\\
&=\sum_{i=1}^k\{[a_1\binom{\eta_1(x_i)}{2}+b\eta_1(x_i)\eta_2(x_i)][-p(\eta_1,x_{(i)},k-1)p(\eta_2,y,l)]\\
&\quad+[a_2\binom{\eta_2(x_i)}{2}+b_2\eta_1(x_i)\eta_2(x_i)][p(\eta_1,x_{(i)},k-1)p(\eta_2,y,l)]\}\\
&\quad+\sum_{i=1}^l\{[a_1\binom{\eta_1(y_i)}{2}+b_1\eta_1(y_i)\eta_2(y_i)][p(\eta_1,x,k)p(\eta_2,y_{(i)},l-1)]\\
&\quad+[a_2\binom{\eta_2(y_i)}{2}+b_2\eta_1(y_i)\eta_2(y_i)][-p(\eta_1,x,k)p(\eta_2,y_{(i)},l-1)]\}
\end{aligned}$$

where $x_{(i)} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) \in \mathbb{Z}^{k-1}$, $y_{(i)} = (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_l) \in \mathbb{Z}^{l-1}$. Furthermore,

$$
\begin{aligned}
&\Omega_r f_{x,y}(\eta) \\
&= \sum_{i=1}^{k} \{ -\frac{1}{2} a_1 p(\eta_1, x^{(i)}, k+1) p(\eta_2, y, l) \\
&+ \frac{1}{2} a_2 p(\eta_1, x_{(i)}, k-1) p(\eta_2, y(x_i, x_i), l+2) \\
&+ (b_2 - b_1) p(\eta_1, x, k) p(\eta_2, y(x_i), l+1) \} \\
&+ \sum_{i=1}^{l} \{ \frac{1}{2} a_1 p(\eta_1, x(y_i, y_i), k+2) p(\eta_2, y_{(i)}, l-1) \\
&- \frac{1}{2} a_2 p(\eta_1, x, k) p(\eta_2, y^{(i)}, l+1) \\
&+ (b_1 - b_2) p(\eta_1, x(y_i), k+1) p(\eta_2, y, l) \}
\end{aligned}
\tag{3.3}
$$

where $x^{(i)} = (x_i, x) \in \mathbb{Z}^{k+1}$, $x(y_i) = (y_i, x) \in \mathbb{Z}^{k+1}$, $x(y_i, y_i) = (y_i, y_i, x) \in \mathbb{Z}^{k+2}$.

By (2.3) Remark we can choose a subsequence of $\varepsilon > 0$ such that for any $k, l \in \mathbb{Z}_+$, $r \in \mathbb{R}^k$, $q \in \mathbb{R}^l$ and $t \geq 0$, the limit of $E^{\varepsilon}_{\mu^{\varepsilon}} p(\eta_t^{(1)}, [\varepsilon^{-1} r], k) \cdot p(\eta_t^{(2)}, [\varepsilon^{-1} q], l)$ exists along this subsequence. Let us denote the limit by $f(r, k, q, l, t)$. On the other hand, it is well known that the simple random walk with time scaling ε^{-2} and space scaling ε^{-1} converges weakly to the Brownian motion. Therefore, by (3.1), (3.2) and (3.3), we have

$$
\begin{aligned}
&f(r_0, k, q_0, l, t) \\
&= \int B(t, (r_0, q_0), (dr, dq)) f(r, k, q, l, 0) \\
&+ \int_0^t ds \int B(t-s, (r_0, q_0), (dr, dq)) \{ \sum_{i=1}^{k} [-\frac{1}{2} a_1 f(r^{(i)}, k+1, q, l, s) \\
&+ \frac{1}{2} a_2 f(r_{(i)}, k-1, q(r_i, r_i), l+2, s) + (b_2 - b_1) f(r, k, q(r_i), l+1, s)] \\
&+ \sum_{i=1}^{l} [\frac{1}{2} a_1 f(r(q_i, q_i), k+2, q_{(i)}, l-1, s) \\
&- \frac{1}{2} a_2 f(r, k, q^{(i)}, l+1, s) + (b_1 - b_2) f(r(q_i), k+1, q, l, s)] \}
\end{aligned}
\tag{3.4}
$$

where $r^{(i)}, r_{(i)}, r(q_i)$ and $r(q_i, q_i)$ could be defined in a similar way as above, $B(t, r, \cdot)$ is the transition probability of the multidimensional independent Brownian motion and $f(r, k, q, l, 0) = \prod_{i=1}^{k} \rho_1(r_i) \prod_{i=1}^{l} \rho_2(q_i)$ by the hypothesis (H_1).

(II). Now, we prove the uniqueness of the solution to equation (3.4). Moreover, the solution possesses the following property: For any $T > 0$, there exists an $M(T) > 0$ such that for any $k, l \in \mathbb{Z}_+, t \geq T, r \in \mathbb{R}^k$ and $q \in \mathbb{R}^l$,

$$
|f(r, k, q, l, t)| \leq M(T)^{k+l}
\tag{3.5}
$$

Suppose f and $\tilde{f}$ are two solutions to equation (3.4) and satisfy (3.5). Set $\varphi = f - \tilde{f}$, then φ is also a solution to equation (3.4) with zero initial value. Set

$$\varphi(t, m) = \sup_{k+l=m, r\in\mathbb{R}^k, q\in\mathbb{R}^l} |\varphi(r, k, q, l, t)|$$

From equation (3.4), it is easy to see that

$$\varphi(t, m) \leq Cm \int_0^t \varphi(s, m+1) ds$$

where $C = (b_1 + b_2) + (a_1 + a_2)/2$. For fixed m, by (3.5) we see that for any $n > 0$ and $t \leq T$,

$$\varphi(t, m) \leq 2C^n m(m+1)\cdots(m+n-1)M(T)^{m+n}t^n/n!$$

Therefore, for any $m > 0$ and $t \in [0, T\wedge(CM(T))^{-1})$, we have

$$\varphi(t, m) = 0. \tag{3.6}$$

If $(CM(T))^{-1} < T$, considering $t' \in (0, (CM(T))^{-1})$ as the initial time of equation (3.4), we can prove similarly that for any $m > 0$ and $t \in [t', T\wedge(CM(T))^{-1} + t')$, (3.6) holds.

By induction, the uniqueness of the solution to equation (3.4) is proved.

(III). By Lemma 2 and (2.2) Remark, the limit $f(r, k, q, l, t)$ satisfies (3.5). However, for the solution $(f_1(r, t), f_2(r, t))$ to equation (1.6), it is easy to verify that $\prod_{i=1}^k f_1(r_i, t)$ $\prod_{i=1}^l f_2(q_i, t)$ satisfies the differential equation (3.4) and (3.5). Therefore, we have

$$f(r, k, q, l, t) = \prod_{i=1}^{k} f_1(r_i, t) \prod_{i=1}^{l} f_2(q_i, t).$$

Consequently, (1.5), (1.6) and (1.7) follows from above.

Corollary: If the conditions of the Theorem are satisfied, then for any $\varphi \in \Phi(\mathbb{R})$ (i.e. the Schwartz space of rapidly decreasing functions on $\mathbb{R}$), the density field satisfies

$$\lim_{\varepsilon\to 0} E^{\varepsilon}_{\mu^\varepsilon}[\varepsilon \sum_{x\in\mathbb{Z}} \varphi(\varepsilon x)\eta_t^{(i)}(x) - \int \varphi(r) f_i(r, t) dr]^2 = 0, i = 1, 2.$$

where $(f_1(r, t), f_2(r, t))$ is the solution to equation (1.6).

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