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ON THREE CLASSICAL PROBLEMS FOR MARKOV CHAINS WITH CONTINUOUS TIME PARAMETERS

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Abstract

For a given transition rate, i.e., a Q-matrix $Q = (q_{ij})$ on a countable state space, the uniqueness of the Q-semigroup $P(t) = (p_{ij}(t))$, the recurrence and the positive recurrence of the corresponding Markov chain are three fundamental and classical problems, treated in many textbooks. As an addition, this paper introduces some practical results motivated from the study of a type of interacting particle systems, reaction diffusion processes. The main results are theorems (1.11), (1.17) and (1.18). Their proofs are quite straightforward.

UNIQUENESS; RECURRENCE; POSITIVE RECURRENCE

1. Statement of the results

Let E be a countable set. Suppose that $(p_{ij}(t))$ is a sub-Markov transition probability matrix having the following properties:

(1.1) *Normalized condition*:

$$p_{ij}(t) \geqq 0, \qquad \sum_j p_{ij}(t) \leqq 1;$$

(1.2) *Chapman–Kolmogorov equation*:

$$p_{ij}(t+s) = \sum_k p_{ik}(t)\, p_{kj}(s), \qquad t, s \geqq 0;$$

(1.3) *Jump condition*:

$$\lim_{t \to 0} p_{ij}(t) = \delta_{ij}, \qquad i, j \in E.$$

It is well known that for such a $(p_{ij}(t))$, we have a Q-matrix $Q = (q_{ij})$ satisfying the
(1.4) *Q-condition*:

$$\lim_{t \to 0} (p_{ij}(t) - \delta_{ij})/t = (1 - \delta_{ij})q_{ij} + q_{ii}\delta_{ij}, \qquad i, j \in E$$

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where

$$0 \leqq q_{ij} < \infty, \quad i \neq j, \quad 0 \leqq q_i \equiv -q_{ii} \leqq \infty, \quad \sum_{j \neq i} q_{ij} \leqq q_i.$$

Because of (1.4), we often call $P(t) = (p_{ij}(t))$ a Q-process.

Throughout this paper, we suppose that the Q-matrix $Q = (q_{ij})$ is totally stable and conservative. That is

$$(1.5) \qquad q_i < \infty, \quad \sum_{j \neq i} q_{ij} = q_i, \qquad i \in E.$$

The first problem of our study is when there is only one Q-process $P(t) = (p_{ij}(t))$ for a given Q-matrix $Q = (q_{ij})$ (then the matrix Q is often called regular). This problem was solved by Feller (1957) and Reuter (1957).

(1.6) *Uniqueness criterion. For a given Q-matrix $Q = (q_{ij})$, the Q-process $(p_{ij}(t))$ is unique if and only if the equation*

$$(1.7) \qquad (\lambda + q_i)u_i = \sum_{j \neq i} q_{ij} u_j, \qquad 0 \leqq u_i \leqq 1, \quad i \in E$$

has only the trivial solution $u_i \equiv 0$ for some (equivalently, for all) $\lambda > 0$.

Certainly, this criterion has many applications. For instance, it gives us a complete answer to the birth–death processes (cf. Corollary (1.16) below). However, it seems hard to apply the above criterion directly to the following examples.

(1.8) *Example*: *Schlögl model.* Let S be a finite set and $E = \mathbb{Z}_+^S$. The model is defined by the following Q-matrix $Q = (q(x, y): x, y \in E)$:

$$q(x,y) = \begin{cases} \lambda_1 \binom{x(u)}{2} + \lambda_4, & \text{if } y = x + e_u, \\ \lambda_2 \binom{x(u)}{3} + \lambda_3 x(u), & \text{if } y = x - e_u, \\ x(u)\, p(u,v), & \text{if } y = x - e_u + e_v, \\ 0, & \text{other } y \neq x \end{cases}$$

$$q(x) = -q(x,x) = \sum_{y \neq x} q(x,y),$$

where $x = (x(u): u \in S)$, $\binom{n}{k}$ is the usual combination, $\lambda_1, \cdots, \lambda_4$ are positive constants, $(p(u, v): u, v \in S)$ is a transition probability matrix on S and e_u is the element in E having value 1 at u and 0 elsewhere.

The Schlögl model is a model of chemical reaction with diffusion in a container. Suppose that the container consists of small vessels. In each vessel $u \in S$, there is a reaction described by a birth–death process. The birth and death rates are given respectively by the first two lines in the definition of $(q(x, y))$ above. Moreover, suppose that between any two vessels u and v, there is a diffusion, with rate given by the third line of the definition. This model was introduced by Schlögl (1972) as a typical model of non-equilibrium systems. See Haken (1983) for related references. See also Chen (1990) for a study of the infinite-dimensional case and for more mathematical references.

(1.9) *Example*: *dual chain of spin system*. Let S be a countable set, and X be the set of all finite subsets of S. For $A \in X$, let $|A|$ denote the number of elements in A. For various concrete models, their Q-matrices $(q(A, B): A, B \in X)$ usually satisfy the following condition:

$$\sum_{B \in X} q(A, B)(|B| - |A|) \leqq C + c|A|, \qquad A \in X \tag{1.10}$$

for some constant $C, c \in \mathbb{R}$. A particular case is that

$$q(A, B) = \sum_{u \in A} c(u) \sum_{F: F \triangle (A \setminus u) = B} p(u, F),$$

where

$$c(u) \geqq 0, \quad \sup_u c(u) < \infty, \quad p(u, A) \geqq 0, \quad \sum_A p(u, A) = 1$$

and

$$\sup_u c(u) \sum_A p(u, A)|A| < \infty.$$

Then (1.10) holds with $C = 0$ and $c = \sup_u c(u) \Sigma_F p(u, F)[|F| - 1]$ (cf. Liggett (1985), Chapter 3, Section 4).

Intuitively, we can interpret the last Markov chain as follows. Let A be the set of sites occupied by particles (finite!). At each site there is at most one particle. Then the process evolves in the following way: each $u \in A$ is removed from A at rate $c(u)$ and is replaced by the set F with probability $p(u, F)$; when an attempt is made to put a point at site u which is already occupied, the two points annihilate one another. The dual chain of the spin system is often used as a dual process of an infinite particle system. This dual approach is one of the main powerful tools in the study of infinite particle systems.

Now, we state our first result.

(1.11) *Theorem. Let* $Q = (q_{ij})$ *be a* Q*-matrix on* E. *Suppose that there exists a sequence* $\{E_n\}_1^\infty$ *and a non-negative function* φ *such that*

$$E_n \uparrow E, \quad \sup_{i \in E_n} q_i < \infty, \quad \lim_{n \to \infty} \inf_{i \notin E_n} \varphi_i = \infty.$$

If in addition

$$\sum_j q_{ij}(\varphi_j - \varphi_i) \leqq c\varphi_i, \qquad i \in E \tag{1.12}$$

holds for some $c \in \mathbb{R}$. *Then the* Q*-process is unique.*

To compare this theorem with the criterion (1.6), we reformulate (1.6) as follows.

(1.13) *Alternative uniqueness criterion*. For a given Q-matrix $Q = (q_{ij})$, the Q-process is unique if and only if the inequality

$$\sum_j q_{ij}(\varphi_j - \varphi_i) \geqq \lambda\varphi_i, \qquad i \in E$$

has no bounded non-constant solution $(\varphi_i : i \in E)$ for some (equivalently, for all) $\lambda > 0$.

Take $E_n = \{i \in E : q_i \leqq n\}$. Then by Theorem (1.11) we have the following result.

(1.14) *Corollary. If there exist a function* $\varphi : \varphi_i \geqq q_i$, $i \in E$ *and a constant* $c \in \mathbb{R}$ *such that* (1.12) *holds, then the Q-process is unique.*

To see these results are practical, for example (1.8), we can either take $\varphi(x) = c[1 + (\Sigma_{u \in S} x(u))^3]$ and apply Corollary (1.14) or take $\varphi(x) = c[1 + \Sigma_u x(u)]$ and apply Theorem (1.11) with $E_n = \{i : i \leqq n\}$, where c is a constant chosen by a simple computation. For example (1.9), simply take $\varphi(A) = c[1 + |A|]$ for a suitable c and apply Theorem (1.11) with $E_n = \{A : |A| \leqq n\}$. For instance, for example (1.8), when $\Sigma_u x(u)$ is large, then (1.12) should hold because the order of the death rate is higher than that of the birth rate. On the other hand, if $\Sigma_u x(u)$ is bounded, then we can choose c large enough so that (1.12) still holds.

Next, we consider a typical case. Let

$$E = \{0, 1, 2, \cdots\} = \mathbb{Z}_+.$$

Suppose that the solution (u_i) to the equation

$$(\lambda + q_i)u_i = \sum_{j \neq i} q_{ij} u_j, \qquad u_0 = 1, \quad i \in E \tag{1.15}$$

is non-decreasing:

$$u_i \uparrow \quad \text{as } i \uparrow$$

then, from criterion (1.6), it is easy to see that the process is unique if and only if $\lim_{i \to \infty} u_i = \infty$. On the other hand, if we take $E_n = \{i \in \mathbb{Z}_+ : i \leqq n\}$, $c = \lambda$ and $\varphi_i = u_i$, $i \in E$, then the hypotheses of Theorem (1.11) are reduced to the condition: $\lim_{i \to \infty} \varphi_i = \lim_{i \to \infty} u_i = \infty$, which is the same as above. Thus, the conditions of Theorem (1.11) are not only sufficient but also necessary for this particular case. This remark plus the next result gives us another view of justifying the power of Theorem (1.11).

(1.16) *Corollary. For the single birth process* $E = \mathbb{Z}_+$,

$$q_{i,i+1} > 0, \quad q_{i,i+k} = 0, \qquad k \geqq 2, \quad i \in E$$

(*but there is no restriction to the death rates*) *the Q-process is unique if and only if* $\Sigma_{k=0}^{\infty} m_k = \infty$, *where*

$$m_k = \sum_{i=0}^{k} F_k^{(i)}/q_{i,i+1}, \qquad k \in E;$$

$$F_k^{(k)} = 1, \quad F_k^{(i)} = \sum_{j=i}^{k-1} q_k^{(j)} F_j^{(i)}/q_{k,k+1}, \qquad 0 \leqq i < k,$$

$$q_k^{(i)} = \sum_{j=0}^{i} q_{kj}, \qquad 0 \leqq i < k, \quad k \in E.$$

Proof. In this case, it was proved in Yan and Chen (1986), Theorem 3, that the solution to (1.15) has the non-decreasing property mentioned above.

Now, we go to the next topic: recurrence. It is well known that for a regular Q, the corresponding Markov chain is recurrent if and only if so is its embedding chain: see Chung (1967). However, we have a more precise formula.

(1.17) *Theorem.*

$$\int_0^\infty p_{ij}^{\min}(t)dt = \sum_{n=0}^{\infty} \bar{p}_{ij}^{(n)}/q_j, \qquad i,j \in E$$

where $\bar{p}_{ij}^{(0)} = \delta_{ij}$, $(\bar{p}_{ij}^{(n)})$ *is the nth power of the matrix* $(\bar{p}_{ij})$:

$$\bar{p}_{ij} = \begin{cases} \delta_{ij}, & \text{if } q_i = 0 \\ (1-\delta_{ij})q_{ij}/q_i, & \text{if } q_i \neq 0, \end{cases}$$

and we use the usual convention: $c/\infty = 0$, $c \neq 0$; $c/0 = \infty$, $c > 0$; $c + \infty = \infty$; $c \times \infty = \infty$, $c > 0$; *and* $0 \times \infty = 0$, $0/0 = 0$.

The last topic is positive recurrence. We call a function $h: E \to \mathbb{R}_+ = [0, \infty)$ compact, if for each $d \in \mathbb{R}_+$, the set $\{i \in E : h_i \leqq d\}$ is finite.

(1.18) *Theorem. Given an irreducible Q-matrix* $Q = (q_{ij})$. *Suppose that there exist a compact function h and constants* $K \geqq 0$, $\eta > 0$ *such that*

$$\sum_j q_{ij}(h_j - h_i) \leqq K - \eta h_i, \qquad i \in E \tag{1.19}$$

then the Markov chain is positive recurrent and hence has a unique stationary distribution.

To apply this theorem to example (1.8), take $h(x) = \Sigma_{u \in S}\, x(u)$ and an arbitrary $\eta > 0$. Then one can find a $K < \infty$ such that the above inequality holds. Hence, the Schlögl model is always ergodic in the finite-dimensional case. As for example (1.9), since $\varnothing$ is an absorbing state, the answer is obvious. Finally, consider the linear growth model:

$$q_{i,i+1} = \lambda i + \delta, \quad q_{i,i-1} = \mu i, \qquad \lambda, \mu, \delta > 0,$$
$$q_{i,j} = 0 \quad \text{for other } j \neq i \pm 1,\ i,j \in \mathbb{Z}_+.$$

It is well known that this model is positive recurrent if and only if $\lambda < \mu$. Recall that this conclusion is usually obtained by studying three series respectively to show the regularity of Q, the recurrence and finally the positive recurrence of the chain. On the other hand, it is obvious that Theorem (1.18) is applicable for the natural choice of $h_i = i (i \in \mathbb{Z}_+)$ if and only if $\lambda < \mu$. Thus, we arrive at precisely the same place directly. Now, the advantage of Theorem (1.18) should be clear.

Most of the above results were studied in Chen (1986b) in Chinese; Theorem (1.11) and Theorem (1.18) were treated for general measurable state $(E, \mathscr{E})$. See also Chen (1986a), (1989). In the context of diffusion processes an analogue of Theorem (1.11) appeared in Stroock and Varadhan (1979). The equality in Theorem (1.17) I learnt from S. W. He (oral communication). A referee pointed out that this formula is well known, and introduced me a to probabilistic argument (due to D. Kendall many years ago). Since it is a nice formula that I have not yet seen in print, and its proof (given later) is quite simple and based on the same approach that is used several times in the paper, I have included this result for completeness. In contrast to Chen (1986b) in this paper we restrict ourselves to the discrete case with a slight extension so that the results are more accessible to a wide range of readers. Indeed, we can prove the above results quite directly. This enables us to teach them in a course on stochastic processes for undergraduate students.

2. Preliminaries

We begin this section by introducing a useful tool, which I learnt from Z. T. Hou, the minimal (non-negative) solution to a non-negative linear equation (Hou and Guo (1978)).

Let E be a countable set, c_{ij}, $b_i \in [0, \infty]$, $i, j \in E$. Consider the equation

$$x_i = \sum_{k \in E} c_{ik} x_k + b_i, \qquad i \in E; \tag{2.1}$$

here we use the conventions $0 \times \infty = 0$, etc., mentioned in the last section.

(2.2) *Existence theorem. Define*

$$x_i^{(0)} \equiv 0, \quad x_i^{(n+1)} = \sum_{k \in E} c_{ik} x_k^{(n)} + b_i, \qquad i \in E, \quad n \geqq 0.$$

Then for every $i \in E$,

$$x_i^{(n)} \uparrow \textit{some } x_i^* \in [0, \infty] \quad \textit{as } n \uparrow \infty,$$

$(x_i^*; i \in E)$ is a solution to (2.1) and is indeed the minimal one in the following sense: for each solution $(x_i : i \in E)$ to (2.1), we have

$$x_i \geqq x_i^*, \qquad i \in E.$$

(2.3) *Corollary. Define*

$$y_i = b_i, \qquad i \in E,$$

$$y_i^{(n+1)} = \sum_{k \in E} c_{ik} y_k^{(n)}, \qquad i \in E, \quad n \geqq 1.$$

Then

$$x_i^* = \sum_{n=1}^{\infty} y_i^{(n)}, \qquad i \in E.$$

(2.4) *Linear combination theorem. Let G be a countable set and c_{ij}, $b_i^{(l)}$, $\alpha_l \in [0, \infty]$, $i, j \in E$, $l \in G$. For each $l \in G$, let $(x_i^{*(l)} : i \in E)$ be the minimal solution to the equation*

$$x_i = \sum_{k \in E} c_{ik} x_k + b_i^{(l)}, \qquad i \in E.$$

Then $(\sum_{l\in G}\alpha_l x_i^{*(l)} : i \in E)$ *is the minimal solution to the equation*:

$$x_i = \sum_{k\in E} c_{i,k} x_k + \sum_{l\in G} \alpha_l b_i^{(l)}, \qquad i \in E.$$

(2.5) *Comparison theorem. Let* $c_{ij}^{(l)}, b_i^{(l)} \in [0,\infty]$, $i, j \in E$, $l = 1, 2$. *Suppose that*

$$c_{ij}^{(1)} \geqq c_{ij}^{(2)}, \quad b_i^{(1)} \geqq b_i^{(2)}, \qquad i, j \in E.$$

Then for every $(x_i^{(1)} : i \in E)$ *which satisfies*

$$x_i \geqq \sum_k c_{ik}^{(1)} x_k + b_i^{(1)}, \qquad i \in E,$$

we have

$$x_i^{(1)} \geqq x_i^{*(2)}, \qquad i \in E,$$

where $(x_i^{*(2)} : i \in E)$ *is the minimal solution to* (2.1) *with coefficients* $(c_{ik}^{(2)})$ *and* $(b_i^{(2)})$.

(2.6) *Monotone convergence theorem. For* $l = 1, 2, \cdots$, *let* $c_{ij}^{(l)}$ *and* $b_i^{(l)}$ *be the same as above and let* $(x_i^{*(l)})$ *denote the corresponding minimal solution. Suppose that*

$$c_{ij}^{(l)} \uparrow c_{ij}, \quad b_i^{(l)} \uparrow b_i \quad \text{as } l \uparrow \infty,\ i, j \in E.$$

Then

$$x_i^{*(l)} \uparrow x_i^*, \quad \text{as } l \uparrow \infty,\ i \in E$$

where $(x_i^* : i \in E)$ *is the minimal solution to* (2.1).

The proofs of the above results are quite elementary. Actually, only the induction and the ordinary monotone convergence theorem are needed, so it is not necessary to present the details here.

Now, we return to our main context. Let

$$p_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)\,dt, \qquad i, j \in E, \quad \lambda > 0.$$

It follows from (1.1)–(1.4) that $(p_{ij}(\lambda))$ satisfies the following conditions.

(2.7) *Normalized condition*

$$p_{ij}(\lambda) \geqq 0, \quad \lambda \sum_j p_{ij}(\lambda) \leqq 1, \qquad i, j \in E, \quad \lambda > 0.$$

(2.8) *Resolvent condition*

$$p_{ij}(\lambda) - p_{ij}(\mu) + (\lambda - \mu) \sum_k p_{ik}(\lambda) p_{kj}(\mu) = 0, \qquad i, j \in E, \quad \lambda, \mu > 0.$$

(2.9) *Jump condition*

$$\lim_{\lambda \uparrow \infty} \lambda p_{ij}(\lambda) = \delta_{ij}, \qquad i, j \in E.$$

(2.10) *Q-condition*

$$\lim_{\lambda\uparrow\infty}\lambda[\lambda p_{ij}(\lambda)-\delta_{ij}]=(1-\delta_{ij})q_{ij}+q_{ii}\delta_{ij},\qquad i,j\in E.$$

The converse implication is also correct. In other words, if $(p_{ij}(\lambda))$ satisfies the conditions (2.7)–(2.10), then there exists uniquely a Q-process $(p_{ij}(t))$ with the given $(p_{ij}(\lambda))$ as its Laplace transform. For a proof of the above facts, see Feller (1957), Reuter (1957), or Gihman and Skorohod (1983). We recall that the advantage of using the Laplace transform is reducing integral equations to the corresponding algebraic equations. Certainly, one may avoid the Laplace transforms, but the proofs given below would become a little more complicated. Based on the above-mentioned one-to-one correspondence between $(p_{ij}(t))$ and its Laplace transform $(p_{ij}(\lambda))$, we also call the latter a Q-process.

(2.11) *Feller's construction. Fix $\lambda>0$ and $j\in E$, let $(p_{ij}^{\min}(\lambda))$ be the minimal solution to the equation*

$$x_i=\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}x_k+\frac{\delta_{ij}}{\lambda+q_i},\qquad i\in E. \tag{2.12}$$

In detail, if we set

$$x_i^{(0)}\equiv 0,$$

$$x_i^{(n+1)}=\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}x_k^{(n)}+\frac{\delta_{ij}}{\lambda+q_i},$$

then

$$x_i^{(n)}\uparrow p_{ij}^{\min}(\lambda)\quad \text{as } n\uparrow\infty,\ i\in E.$$

This $P^{\min}(\lambda)=(p_{ij}^{\min}(\lambda))$ is a Q-process and is the minimal one. In other words, for any Q-process $(p_{ij}(\lambda))$, we have

$$p_{ij}(\lambda)\geqq p_{ij}^{\min}(\lambda),\qquad i,j\in E,\quad \lambda>0.$$

Moreover, for fixed $\lambda>0$ and $i\in E$, $(p_{ij}^{\min}(\lambda): j\in E)$ is also the minimal solution to the equation

$$y_j=\sum_{k\neq j}y_k\frac{q_{kj}}{\lambda+q_j}+\frac{\delta_{ij}}{\lambda+q_j},\qquad j\in E. \tag{2.13}$$

Proof. For the reader's convenience, we sketch the proof here. By using the equation

$$p_{ij}^{\min}(\lambda)=\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}p_{kj}^{\min}(\lambda)+\frac{\delta_{ij}}{\lambda+q_i},\qquad i\in E$$

the conditions (2.7), (2.9) and (2.10) are easy to check. Put

$$\tilde{p}_{ij}^{(1)}(\lambda) = \frac{\delta_{ij}}{\lambda + q_i}$$

$$\tilde{p}_{ij}^{(n+1)} = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} \tilde{p}_{kj}^{(n)}(\lambda), \qquad n \geqq 1, \quad i, j \in E.$$

Then the condition (2.8) follows from Corollary (2.3):

$$\sum_{n=1}^{\infty} \tilde{p}_{ij}^{(n)}(\lambda) = p_{ij}^{\min}(\lambda)$$

and the equality

$$\tilde{p}_{ij}^{(n)}(\lambda) - \tilde{p}_{ij}^{(n)}(\mu) = (\mu - \lambda) \sum_{m=1}^{n} \sum_{k} \tilde{p}_{ik}^{(m)}(\lambda) \tilde{p}_{kj}^{(n+1-m)}$$

(2.14)

$$\lambda, \mu > 0, \quad i, j \in E, \quad n \geqq 1.$$

To prove (2.14), simply use induction on n.

Next, by using induction step by step, it is easy to show that (2.12) and (2.13) give us the same minimal solution.

Finally, the fact that $(p_{ij}^{\min}(\lambda))$ is minimal follows from Theorem (2.2) and the following result.

(2.15) *Lemma. Let* $(p_{ij}(\lambda))$ *be a Q-process, then*

$$p_{ij}(\lambda) \geqq \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} p_{kj}(\lambda) + \frac{\delta_{ij}}{\lambda + q_i}, \qquad \lambda > 0, \quad i, j \in E.$$

Proof. By (2.8), we have

$$-\mu p_{ij}(\mu) + \mu p_{ii}(\mu) \lambda p_{ij}(\lambda) + \sum_{k \neq i} \mu p_{ik}(\mu) \lambda p_{kj}(\lambda)$$

$$+ \mu(1 - \mu p_{ii}(\mu)) p_{ij}(\lambda) = \sum_{k \neq i} \mu^2 p_{ik}(\mu) p_{kj}(\lambda).$$

Letting $\mu \to \infty$ and using Fatou's lemma, we obtain

$$-\delta_{ij} + \lambda p_{ij}(\lambda) + q_i p_{ij}(\lambda)$$

$$\geqq \sum_{k \neq i} q_{ik} p_{kj}(\lambda),$$

since

$$\limsup_{\mu \to \infty} \sum_{k \neq i} \mu p_{ik}(\mu) \lambda p_{kj}(\lambda)$$

$$\leqq \limsup_{\mu \to \infty} \sum_{k \neq i} \mu p_{ik}(\mu)$$

$$= \limsup_{\mu \to \infty} (1 - \mu p_{ii}(\mu)) = 0.$$

3. Proofs of the results

(3.1) *Proof of Theorem* (1.11).

(i) We first consider the trivial case that $\sup_i q_i = C < \infty$. By (2.13), for every non-negative function f, we have

$$\sum_j p_{ij}^{\min}(\lambda) f_j = \sum_k p_{ik}^{\min}(\lambda) \sum_{j \neq k} \frac{q_{kj}}{\lambda + q_j} f_j + \frac{f_i}{\lambda + q_i}.$$

In particular, taking $f_j = \lambda + q_j$, we get

$$\lambda \sum_j p_{ij}^{\min}(\lambda) + \sum_j p_{ij}^{\min}(\lambda) q_j = \sum_k p_{ik}^{\min}(\lambda) q_k + 1.$$

Since

$$\sum_j p_{ij}^{\min}(\lambda) q_j \leqq C/\lambda < \infty,$$

we have

$$\lambda \sum_j p_{ij}^{\min}(\lambda) = 1, \qquad \lambda > 0, \quad i \in E.$$

By Feller's construction (2.11), this certainly implies the uniqueness of Q-processes.

(ii) For the general case, let

$$q_{ij}^{(n)} = I_{E_n}(i) q_{ij}, \quad q_i^{(n)} = \sum_{j \neq i} q_{ij}^{(n)}, \qquad i, j \in E.$$

Then for each n, $\sup_i q_i^{(n)} < \infty$. Because of (i), we can let $(p_{ij}^{(n)}(\lambda))$ denote the Q-process determined by $(q_{ij}^{(n)})$. By the condition (1.12), we have

$$\sum_j q_{ij}^{(n)}(\varphi_j - \varphi_i) \leqq c\varphi_i, \qquad i \in E \tag{3.2}$$

for some $c > 0$. By Feller's construction (2.11) and linear combination theorem (2.4), it follows that for fixed $\lambda > 0$, $(\sum_j p_{ij}^{(n)}(\lambda)\varphi_j : i \in E)$ is the minimal solution to the equation

$$x_i = \sum_{k \neq i} \frac{q_{ik}^{(n)}}{\lambda + q_i^{(n)}} x_k + \frac{\varphi_i}{\lambda + q_i^{(n)}}, \qquad i \in E.$$

On the other hand, the condition (3.2) gives us

$$\frac{\varphi_i}{\lambda - c} \geqq \sum_{k \neq i} \frac{q_{ik}^{(n)}}{\lambda + q_i^{(n)}} \frac{\varphi_k}{\lambda - c} + \frac{\varphi_i}{\lambda + q_i^{(n)}}, \qquad i \in E, \quad \lambda > c.$$

Hence, by the comparison theorem (2.5),

$$\sum_j p_{ij}^{(n)}(\lambda)\varphi_j \leqq \frac{\varphi_i}{\lambda - c} \qquad \lambda > c, \quad i \in E, \quad n \geqq 1. \tag{3.3}$$

(iii) For $i \in E_n$, by (2.11) we have

$$\sum_{j\in E_n} p_{ij}^{\min}(\lambda) = \sum_{k\neq i} \frac{q_{ik}}{\lambda+q_i} \sum_{j\in E_n} p_{kj}^{\min}(\lambda) + \sum_{j\in E_n} \frac{\delta_{ij}}{\lambda+q_i}$$

(3.4)

$$= \sum_{k\neq i} \frac{q_{ik}^{(n)}}{\lambda+q_i^{(n)}} \sum_{j\in E_n} p_{kj}^{\min}(\lambda) + \sum_{j\in E_n} \frac{\delta_{ij}}{\lambda+q_i^{(n)}}.$$

On the other hand, if $i\notin E_n$, then

$$\sum_{j\in E_n} p_{ij}^{\min}(\lambda) \geqq 0 = \text{RHS of (3.4)}.$$

Hence, for all $i\in E$, we have

$$\sum_{j\in E_n} p_{ij}^{\min}(\lambda) \geqq \text{RHS of (3.4)}.$$

By Theorems (2.11) and (2.6), we obtain

$$\sum_{j\in E_n} p_{ij}^{\min}(\lambda) \geqq \sum_{j\in E_n} p_{ij}^{(n)}(\lambda), \qquad i\in E, \quad \lambda>0, \quad n\geqq 1.$$

(iv) Combining the above facts, we arrive at

$$\begin{aligned}\lambda \sum_{j\in E_n} p_{ij}^{\min}(\lambda) &\geqq \lambda \sum_{j\in E_n} p_{ij}^{(n)}(\lambda)\\ &= 1-\lambda \sum_{j\notin E_n} p_{ij}^{(n)}(\lambda)\\ &\geqq 1-\lambda\varphi_i \Big/ \Big[(\lambda-c)\inf_{j\notin E_n}\varphi_j\Big], \qquad \lambda>c.\end{aligned}$$

Finally, by (2.8), it is easy to see that the above equality indeed holds for all $\lambda>0$.

(3.5) *Proof of Theorem* (1.17). For simplicity we omit the superscript 'min' from now on. Note that for fixed j and $\lambda>0$, $(p_{ij}(\lambda))$ is the minimal solution to

$$x_i = \sum_{k\neq i} \frac{q_{ik}}{\lambda+q_i} x_k + \frac{\delta_{ij}}{\lambda+q_i}, \qquad i\in E.$$

On the other hand,

$$\frac{q_{ij}}{\lambda+q_i} \uparrow \frac{q_{ij}}{q_i}, \quad \frac{\delta_{ij}}{\lambda+q_i} \uparrow \frac{\delta_{ij}}{q_i},$$

$$p_{ij}(\lambda) \uparrow \int_0^\infty p_{ij}(t)dt, \quad \text{as } \lambda \downarrow 0,\ i,j\in E.$$

By the monotone convergence theorem (2.6), $(\int_0^\infty p_{ij}(t)dt : i\in E)$ is the minimal solution to the equation

$$x_i = \sum_{k\neq i} \frac{q_{ik}}{q_i} x_k + \frac{\delta_{ij}}{q_i}, \qquad i\in E.$$

Next,

$$x_i^{(1)} = \frac{\delta_{ij}}{q_i} = \frac{\bar{p}_{ij}^{(0)}}{q_i}, \qquad i \in E.$$

Suppose that

$$x_i^{(n)} = \sum_{m=0}^{n-1} \bar{p}_{ij}^{(m)}/q_j, \qquad i \in E$$

for some $i \in E$. Then

$$\begin{aligned} x_i^{(n+1)} &= \sum_{k \neq i} \frac{q_{ik}}{q_i} x_k^{(n)} + \frac{\delta_{ij}}{q_i} \\ &= I_{[q_i \neq 0]} \sum_k \bar{p}_{ik}^{(1)} \sum_{m=0}^{n-1} \bar{p}_{kj}^{(m)}/q_j + \bar{p}_{ij}^{(0)}/q_j. \end{aligned} \tag{3.6}$$

and so

$$\text{RHS of (3.6)} = \sum_{m=0}^{n} \bar{p}_{ij}^{(m)}/q_j$$

when $q_i \neq 0$. Otherwise, $q_i = 0$,

$$\bar{p}_{ij}^{(0)}/q_j = \delta_{ij}/q_j = \begin{cases} \infty, & \text{if } j = i \\ 0, & \text{if } j \neq i, \end{cases}$$

$$\sum_{m=0}^{n} \bar{p}_{ij}^{(m)}/q_j = (n+1)\delta_{ij}/q_j = \begin{cases} \infty, & \text{if } j = i \\ 0, & \text{if } j \neq i. \end{cases}$$

Hence we still have

$$\text{RHS of (3.6)} = \sum_{m=0}^{n} \bar{p}_{ij}^{(m)}/q_j.$$

Thus, we have proved that $x_i^{(n)} = \Sigma_{m=0}^{n-1} \bar{p}_{ij}^{(m)}/q_j$ for all $i \in E$ and $n \geqq 1$. By Theorem (2.2),

$$\int_0^\infty p_{ij}(t)dt = \lim_{n\to\infty} x_i^{(n)} = \lim_{n\to\infty} \sum_{m=0}^{n} \bar{p}_{ij}^{(m)}/q_j = \sum_{n=0}^{\infty} \bar{p}_{ij}^{(n)}/q_j, \quad i \in E.$$

(3.7) *Proof of Theorem* (1.18).

(i) Since h is compact, $E_n = \{i : h_i \leqq n\}$ is a finite set, we must have

$$\sup_{i \in E_n} q_i < \infty \quad \text{and} \quad \lim_{n\to\infty} \inf_{i \notin E_n} h_i = \infty.$$

Thus, an application of Theorem (1.11) with $h = \varphi$ tells us that the Q-process is unique.

(ii) By induction, it is easy to check the following fact: the Laplace transform of the minimal solution to the integral equation

$$p_{ij}(t) = \sum_{k \neq i} \int_0^t \exp(-q_i(t-s)) q_{ik} p_{kj}(s) ds + \delta_{ij} \exp(-q_i t), \qquad i \in E$$

is just the minimal solution to (2.12). Thus, by the linear combination theorem (2.4), for fixed $t \geqq 0$, $(\Sigma_j p_{ij}(t)h_j : i \in E)$ is the minimal solution to the equation

$$x_i(t) = \sum_{k \neq i} \int_0^t \exp(-q_i(t-s))q_{ik}x_k(s)ds + \exp(-q_i t)h_i, \qquad i \in E.$$

Now, by the comparison theorem (2.5), the condition (1.19) gives us

$$\sum_j p_{ij}(t)h_j \leqq e^{Kt} - 1 + e^{-\eta t}h_i, \qquad i \in E, \quad t \geqq 0.$$

(iii) Fix $t > 0$ and put

$$C = e^{Kt} - 1 \geqq 0, \qquad c = e^{-\eta t} < 1.$$

By (ii), we have

$$\sum_j p_{ij}(t)h_j \leqq C + ch_i, \qquad i \in E,$$

and so

$$\sum_j p_{ij}(nt)h_j \leqq C(1 + c + \cdots + c^{n-1}) + c^n h_i$$

$$\leqq \frac{C}{1-c} + c^n h_i \leqq \frac{C}{1-c} + ch_i, \qquad i \in E.$$

Next, fix $i \in E$ and set

$$\mu_j^{(n)} = p_{ij}(nt).$$

It will be proved later that the family $\{\mu^{(n)} : n \geqq 1\}$ is tight. Thus, we may assume that

$$\mu_j^{(nk)} \Rightarrow \mu_j \quad \text{as } k \to \infty, j \in E$$

and $\Sigma_j \mu_j = 1$. On the other hand, the chain $(p_{ij}(t))$ is irreducible and hence aperiodic, the limit

$$\lim_{t \to \infty} p_{ij}(t)$$

always exists, which is positive if and only if the chain is positive recurrent.

(iv) To prove the tightness of $\{\mu^{(n)} : n \geqq 1\}$, note that

$$\sum_j \mu_j^{(n)} h_j \leqq D < \infty, \qquad n \geqq 1.$$

For any $\varepsilon > 0$, take

$$K_\varepsilon = \{i \in E : h_i \leqq D/\varepsilon\}.$$

Then K_ε is compact and

$$\sup_n \mu^{(n)}(K_\varepsilon^c) = \sup_n \sum_{j \notin K_\varepsilon} \mu_j^{(n)}$$

$$\leqq \frac{\varepsilon}{D} \sup_n \sum_j \mu_j^{(n)} h_j \leqq \varepsilon.$$

The last argument I learnt from Dobrushin (1970).

(3.8) *Corollary. Under the hypothesis of Theorem* (1.18), *the stationary distribution* $(\mu_i : i \in E)$ *has the property*:

$$\sum_j \mu_j h_j \leqq K/\eta.$$

Proof. As we have seen from the above,

$$\sum_j p_{ij}(nt) h_j \leqq C/(1-c) + c^n h_i, \qquad i \in E, \quad n \geqq 1.$$

Letting $n \to \infty$ and using Fatou's lemma, we obtain

$$\sum_j \mu_j h_j \leqq C/(1-c) = (\exp[Kt] - 1)/(1 - \exp[-\eta t]).$$

Now, the conclusion follows by setting $t \downarrow 0$.

To prove the equivalence of (1.6) and (1.13), we need two lemmas.

(3.9) *Lemma. Let* $c_{ij}, b_i \in [0, 1]$ *satisfy*

$$\sum_j c_{ij} + b_i \leqq 1, \qquad i \in E.$$

Define

$$u_i^{(0)} = 1,$$

$$u_i^{(n+1)} = \sum_k c_{ik} u_k^{(n)} + b_i, \qquad i \in E, \quad n \geqq 0.$$

Then

$$u_i^{(n)} \downarrow \textit{some } \bar{u}_i, \qquad i \in E$$

and $(\bar{u}_i : i \in E)$ *is the maximal solution to the equation*

$$u_i = \sum_k c_{ik} u_k + b_i, \qquad 0 \leqq u_i \leqq 1, \quad i \in E.$$

Proof. By induction and the dominated convergence theorem.

(3.10) *Comparison lemma. Let* $c_{ij}^{(l)}, b_i^{(l)}$ *be the same as above*, $l = 1, 2$. *Suppose that*

$$c_{ij}^{(1)} \geqq c_{ij}^{(2)}, \quad b_i^{(1)} \geqq b_i^{(2)}, \qquad i, j \in E.$$

Then for every $(u_i^{(2)} : i \in E)$ *satisfying*

$$u_i \leqq \sum_j c_{ij}^{(2)} u_j + b_i^{(2)}, \qquad 0 \leqq u_i \leqq 1, \quad i \in E \tag{3.11}$$

we have

$$\bar{u}_i^{(1)} \geqq u_i^{(2)}, \qquad i \in E$$

where $(\bar{u}_i^{(1)} : i \in E)$ *is the maximal solution to the equation*

(3.12) $$u_i = \sum_i c_{ij}^{(1)} u_j + b_i^{(1)}, \qquad 0 \leqq u_i \leqq 1, \quad i \in E.$$

Furthermore, if $c_{ij}^{(1)} = c_{ij}^{(2)}$ *and* $b_i^{(1)} = b_i^{(2)}$ *for all* $i, j \in E$, *then* (3.11) *has no non-trivial solution if and only if so does* (3.12).

Proof. The first assertion follows from Lemma (3.9) and induction. Then the second assertion follows immediately.

(3.13) *Proof of the equivalence of* (1.6) *and* (1.13). Let $(\varphi_i : i \in E)$ be bounded and satisfy

(3.14) $$\sum_j q_{ij}(\varphi_j - \varphi_i) \geqq \lambda \varphi_i, \qquad i \in E$$

for some $\lambda > 0$. If $\inf_k \varphi_k < 0$, we may replace φ_i with $\varphi_i - \inf_k \varphi_k$ and keep the inequality (3.14). Thus, we can assume that (φ_i) is non-negative solution, we may also assume that it is bounded by 1 from the above. Therefore the above-mentioned equivalence follows from Lemma (3.10) immediately.

(3.15) *Remark.* Let $c_{ij}, b_i \in [0, 1]$ satisfy

$$\sum_j c_{ij} + b_i \leqq 1, \qquad i \in E.$$

Denote by $(x_i^* : i \in E)$ the minimal solution to the equation

$$x_i = \sum_j c_{ij} x_j + b_i, \qquad i \in E.$$

Then $(1 - x_i^* : i \in E)$ is the maximal solution to the equation

$$u_i = \sum_j c_{ij} u_j + (1 - b_i), \qquad 0 \leqq u_i \leqq 1, \quad i \in E.$$

From this point of view, the maximal solution is a dual of the minimal solution. Keeping this idea in mind, we can get an alternative proof of Theorem (1.11) by using Lemma (3.10) plus Criterion (1.6). But such a proof is more indirect than the one given above.

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