Journal of Statistical Physics, Vol. 58, Nos. 5/6, 1990

Ergodic Theorems for Reaction-Diffusion Processes

Mu-Fa Chen[1]

Received April 11, 1989; revised received September 26, 1989

New sufficient conditions are given for the ergodicity of reaction-diffusion processes which improve both Neuhauser's recent result and the present author's previous result. In the main criterion; contrary to the previous ones, the pure birth rate of the reaction plays a critical role. To do this, a new but natural coupling is introduced. It is proved that this coupling is the best one in some sense. One of the main results says that the reaction-diffusion processes are ergodic for all large enough pure birth rates.

KEY WORDS: Ergodic theorems; reaction-diffusion processes; coupling; Kantorovich probability distance.

1. INTRODUCTION

The reaction-diffusion processes considered in this paper are continuous-time Markov processes with state space $E=\{\eta\colon \mathbb{Z}^d \to \mathbb{Z}_+ = \{0, 1, 2,...\}\}$. The processes evolve in the following way:

(i) At rate $b(\eta(x))$ a particle is born at x.

(ii) At rate $a(\eta(x))$ a particle at x dies.

(iii) At rate $\eta(x)\, p(x, y)$ a particle jumps from x to y.

Here $p(x, y)$ is a transition probability on $\mathbb{Z}^d$. The formal generator is

$$\begin{aligned}\Omega f(\eta) = \sum_x \{&b(\eta(x))[f(\eta+e_x)-f(\eta)]\\ &+a(\eta(x))[f(\eta-e_x)-f(\eta)]\\ &+\sum_y \eta(x)\, p(x, y)[f(\eta-e_x+e_y)-f(\eta)]\}\end{aligned} \tag{1.1}$$

[1] Department of Mathematics, Beijing Normal University, Beijing 100875, China.

0022-4715/90/0300-0939$06.00/0

Here the sums are over all x and y in $\mathbb{Z}^d$, and $e_x \in E$ has $e_x(x)=1$ and $e_x(y)=0$ for $y \neq x$.

Three concrete examples are:

Linear growth model: $b(k)=\beta_0+\beta_1 k$, $a(k)=\delta_1 k$.

Schlögl's first model: $b(k)=\beta_0+\beta_1 k$, $a(k)=\delta_1 k+\delta_2 k(k-1)$.

Schlögl's second model:

$$b(k)=\beta_0+\beta_2 k(k-1),\ a(k)=\delta_1 k+\delta_3 k(k-1)(k-2).$$

Here the coefficients are all positive.

I assume the following hypothesis (H):

(H). $p(x, y)$ is translation invariant in $\mathbb{Z}^d$, $p(x,x)=0$, and

$$b(k)=\sum_{j=0}^{m} \beta_j k^{(j)}, \qquad a(k)=\sum_{j=1}^{m+1} \delta_j k^{(j)}$$

where $k^{(j)}=k(k-1)\cdots(k-j+1)$, the coefficients β_j and δ_j are nonnegative, $m \geqslant 1$, and β_0, δ_1, $\delta_{m+1}>0$.

Actually, one can allow $\beta_0=0$. But the proof will become much simpler, so I will not consider this situation.

For these systems, the Markov processes were constructed in ref. 1. Since the rates are unbounded, the processes have to be constructed on a smaller state space

$$E_0=\left\{\eta \in E: \sum_x \eta(x)\,\alpha(x)<\infty\right\}$$

where $\alpha(\cdot)$ is a summable positive sequence such that

$$\sum_y p(x, y)\,\alpha(y) \leqslant M\alpha(x), \qquad x \in \mathbb{Z}^d$$

for some M. The uniqueness of the processes was not discussed in ref. 1, but, as X. G. Zheng pointed out to me, it is indeed a straightforward consequence of the construction plus some estimates of higher order moments. I will not discuss the details here.

As for the stationary distributions of the processes, some general existence and uniqueness results were presented in ref. 2 in Chinese. An English version with some improvements is given in ref. 5. Applying the result given in ref. 2, Chapter 14, to the present case, if the condition

$$c+M-1<0$$

holds, where

$$c=\sup_{k \geqslant 0,\, l \geqslant 1}\ [b(k+l)-b(k)-a(k+l)+a(k)]/l$$

then the process is ergodic. In the present case, $p(x, y)$ is translation invariant, so the constant M can be chosen as close to 1 as desired. Hence the condition

$$c < 0 \tag{1.2}$$

is sufficient for the ergodicity. For the linear growth model, the condition (1.2) becomes $\beta_1 < \delta_1$, which cannot be improved anymore.[8] However, this condition is too strong if the reaction is nonlinear. For example, for the first Schlögl model, (1.2) is just

$$\beta_1 < \delta_1 \tag{1.3}$$

For the second Schlögl model, we have a solution to (1.2) as follows:

$$\delta_1 > \beta_2 + \frac{3}{4}\delta_3 + \frac{\beta_2^2}{3\delta_3} \quad (\geqslant 2\beta_2) \tag{1.4}$$

Recently, Ding *et al.*[7] proved the ergodicity of reversible reaction-diffusion processes (i.e., $\delta_i = \alpha\beta_{i-1}$, $1 \leqslant i \leqslant m+1$, for some $\alpha > 0$) under some reasonable hypotheses. By using ideas from that paper, Neuhauser[10] has improved the condition (1.2) in the general case (not necessarily reversible). She used a metric

$$\rho(k, l) = \sum_{0 \leqslant j \leqslant |k-l|-1} u_j \quad \text{on} \quad \mathbb{Z}^d$$

where $1 \geqslant u_j \geqslant \varepsilon > 0$ for some $\varepsilon > 0$ and all $j \in \mathbb{Z}_+$, instead of the ordinary metric $|k-l|$. The latter was used in my original proof of estimating the Kantorovich probability distance (in refs. 2, 4, and 5 it was called the KRW distance). Since $\rho(k, l)$ is equivalent to the ordinary metric, Neuhauser's theorem is also the ergodicity in the same Kantorovich distance (cf. Remark 2.2).

In this paper, some new sufficient conditions for the ergodicity are proposed. The main result is Theorem 1.1, in which a metric not necessarily equivalent to the above one is used. More precisely, I require $u_j > 0$ for all $j \geqslant 0$, but not $u_j \geqslant \varepsilon > 0$ as above; the pure birth rate β_0 appears in the formulation of the criterion of Theorem 1.1. This is an essentially new point. Actually, I will prove that the reaction-diffusion processes are ergodic for all large enough β_0 (Theorem 4.8). The reason I can do this is that I find a new but quite natural coupling and, as is seen in Section 3, it is the best choice in some sense.

To state the main result, I need some notations. Define

$$u_0=1, \qquad u_1=u_1(\varepsilon)=\big(\inf_{k\geqslant 0}[b(k)+a(k+1)-\varepsilon]/[a(k)+b(k+1)+\varepsilon]\big)\vee 0$$

$$u_l=u_l(\varepsilon)=\left\{\inf_{k\geqslant 0}\left[(b(k)\vee a(k+l)+l)u_{l-1}+(b(k)\wedge a(k+l))u_{l-2}\right.\right.$$

$$\left.\left.-l-\varepsilon\sum_{j=0}^{l-1}u_j\right]\Big/[a(k)+b(k+l)+\varepsilon]\right\}\vee 0, \qquad l\geqslant 2,$$

where $a\vee b=\max\{a,b\}$, $a\wedge b=\min\{a,b\}$.

Theorem 1.1 (*u-criterion*). Under (H), if there exists an $\varepsilon>0$ such that $u_l(\varepsilon)>0$ for all $l\geqslant 0$, then the process is ergodic.

This theorem will be proved in the next section. In Section 3, I will introduce some other criteria and compare them with the above criterion and the original condition (1.2). In the last section, I apply the criteria to the Schlögl models. I study mainly for the first model when the criteria are or are not available. Different versions of these criteria and a comparison result for different processes are presented. Having the ergodicity of the reversible processes in mind, we know that our main criterion is still not at the final level. This may be due to the limitation of the coupling technique. It is worth mentioning that the ideas of this paper can be used directly for the study of successful couplings of other Markov processes.

2. PROOF OF THEOREM 1.1

I begin this section with a simple result.

Lemma 2.1. Under (H), the sequence $\{u_l\}$ is decreasing and bounded by 1 from above.

Proof. Use induction. Since the degree of $a(k)$ is higher than that of $b(k)$, we have

$$u_1\leqslant \lim_{k\to\infty}\frac{b(k)+a(k+1)-\varepsilon}{a(k)+b(k+1)+\varepsilon}=1=u_0$$

Suppose that $0\leqslant u_{l-1}\leqslant u_{l-2}\leqslant \cdots \leqslant 1$; then

$$u_l\leqslant \lim_{k\to\infty}\frac{b(k)\vee a(k+l)+l}{a(k)+b(k+l)+\varepsilon}u_{l-1}=u_{l-1}\leqslant 1 \qquad \text{QED}$$

Lemma 2.2 (*Estimates of Moments*). Under (H), for every $m \geq 1$, we have:

(i) $\mathbb{E}^\eta(\eta_t(x)^m) < \infty$, $t \geq 0$, $\eta \in E_0$, $x \in \mathbb{Z}^d$.

(ii) There exists a decreasing function φ_m: $(0, \infty) \to [0, \infty)$ such that

$$\mathbb{E}^\eta(\eta_t(x)^m) \leq \varphi_m(t) \qquad \text{for all} \quad t > 0 \quad \text{and} \quad \eta \in E_0^s$$

where $\mathbb{E}^\eta$ indicates the expected value for the process η_t starting from η and

$$E_0^s = \{\eta \in E_0 : \eta(x) = \eta(0) \text{ for all } x \in \mathbb{Z}^d\}$$

Proof. The first assertion comes as no surprise because the degree of the death rate is higher than the degree of the birth rate. At least when $m = 1$, it was proved in ref. 1, but the same proof works for the general case as well. The second assertion is due to Ding *et al.*,[7] for which we require that the degree of the death rate is at least two. QED

Now, we split the proof of Theorem (1.1) into five steps.

(a) First, consider the finite-dimensional case. Let S be a finite additive group. Suppose that $(p(x, y): x, y \in S)$ is a translation-invariant transition probability:

$$p(x+z, y+z) = p(x, y) \qquad \text{for all} \quad x, y, z \in S$$

By using S instead of $\mathbb{Z}^d$, one can define a generator as in (1.1). I introduce the following coupling for this Markov chain. For the diffusion part, throughout this paper, I couple the process in the following way:

$$\begin{aligned}
(\eta, \zeta) &\to (\eta - e_x + e_y, \zeta - e_x + e_y) && \text{at rate } \eta(x) \wedge \zeta(x)\, p(x, y) \\
&\to (\eta - e_x + e_y, \zeta) && \text{at rate } (\eta(x) - \zeta(x))^+\, p(x, y) \\
&\to (\eta, \zeta - e_x + e_y) && \text{at rate } (\zeta(x) - \eta(x))^+\, p(x, y)
\end{aligned}$$

Notice that whenever simultaneous jumps at rate $A \wedge B$ (say) occur, we automatically get two individual jumps at rates $(A-B)^+$ and $(B-A)^+$, respectively. Thus, in what follows I will write down only the first rate and omit the others for simplicity.

For the reaction part, at each $x \in \mathbb{Z}^d$, I couple the process in the following way:

(i) If $\eta(x) = \zeta(x)$, the two marginal processes are made to evolve at exact the same rates.

(ii) If $|\eta(x)-\zeta(x)|=1$, they are made to jump independently. For example,

$$(\eta, \zeta) \to (\eta+e_x, \zeta) \quad \text{at rate } b(\eta(x))$$
$$\to (\eta, \zeta+e_x) \quad \text{at rate } b(\zeta(x))$$

and so on.

(iii) Let $\eta(x)<\zeta(x)$. If $|\eta(x)-\zeta(x)|\geqslant 2$, choose

$$\begin{aligned}(\eta, \zeta) &\to (\eta+e_x, \zeta-e_x) && \text{at rate } b(\eta(x)) \wedge a(\zeta(x))\\ &\to (\eta-e_x, \zeta) && \text{at rate } a(\eta(x))\\ &\to (\eta, \zeta+e_x) && \text{at rate } b(\zeta(x))\end{aligned}$$

Finally, I use $\tilde{\Omega}$ to denote this coupling operator. It is a generator and clearly order-preserving.[3]

(b) Next, I make some computations. Let $F(k)=\sum_{0\leqslant j\leqslant k-1} u_j$. Then for $\xi(x)=\zeta(x)-\eta(x)\geqslant 0$, $x\in S$, we have

$$\begin{aligned}\tilde{\Omega}F(\xi(x)) = \{&-b(\eta(x))u_{\xi(x)-1}+a(\eta(x))u_{\xi(x)}\\ &+b(\zeta(x))u_{\xi(x)}-a(\zeta(x))u_{\xi(x)-1}\}\, I_{\xi(x)=1}\\ &+\{-[b(\eta(x))\wedge a(\zeta(x))](u_{\xi(x)-2}+u_{\xi(x)-1})\\ &-[b(\eta(x))-a(\zeta(x))]^+ u_{\xi(x)-1}\\ &-[a(\zeta(x))-b(\eta(x))]^+ u_{\xi(x)-1}\\ &+a(\eta(x))u_{\xi(u)}+b(\zeta(x))u_{\xi(x)}\}\, I_{\xi(x)\geqslant 2}\\ &+\sum_y \xi(y)\, p(y,x)u_{\xi(u)}-\xi(x)\sum_y p(x,y)u_{\xi(x)-1}\end{aligned}$$

where $I_{\xi(x)=1}$ is the indicator of the set $[\xi: \xi(x)=1]$. Collecting terms, we obtain

$$\begin{aligned}\tilde{\Omega}F(\xi(x)) = \{&[a(\eta(x))+b(\zeta(x))]u_{\xi(x)}\\ &-[b(\eta(x))+a(\zeta(x))]u_{\xi(x)-1}\}\, I_{\xi(x)=1}\\ &+\{[a(\eta(x))+b(\zeta(x))]u_{\xi(x)}\\ &-[b(\eta(x))\vee a(\zeta(x))]u_{\xi(x)-1}\\ &-[b(\eta(x))\wedge a(\zeta(x))]u_{\xi(x)-2}\}\, I_{\xi(x)\geqslant 2}\\ &-\xi(x)u_{\xi(x)-1}+\sum_y \xi(y)\, p(y,x)u_{\xi(x)}\end{aligned}$$

Note that we can combine $-\xi(x)u_{\xi(x)-1}$ into the reaction part, so we can estimate this by

$$\tilde{\Omega}F(\xi(x)) \leqslant -\varepsilon F(\xi(x)) - \xi(x) + \sum_y \xi(y)\, p(y, x)$$

As mentioned before, the coupling jump process $\tilde{P}(t)$ with generator $\tilde{\Omega}$ is unique, so we have

$$\frac{d}{dt}\tilde{P}(t)\, F(\xi(x)) = \tilde{P}(t)\, \tilde{\Omega}F(\xi(x))$$

Thus, for any initial $\eta \leqslant \zeta$, $\eta, \zeta \in E^s$,

$$E^s = \{\eta \in \mathbb{Z}_+^S : \eta(x) = \eta(0),\ x \in S\}$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\tilde{\mathbb{E}}^{(\eta,\zeta)} F(\xi_t(x)) \\
&\qquad \leqslant -\varepsilon \tilde{\mathbb{E}}^{(\eta,\zeta)} F(\xi_t(x)) - \tilde{\mathbb{E}}^{(\eta,\zeta)} \xi_t(x) + \tilde{\mathbb{E}}^{(\eta,\zeta)} \sum_y \xi_t(y)\, p(y, x) \\
&\qquad = -\varepsilon \tilde{\mathbb{E}}^{(\eta,\zeta)} F(\xi_t(x)), \qquad x \in S, \quad t \geqslant 0
\end{aligned}$$

where $\xi_t = \zeta_t - \eta_t$. Here I use attractiveness, $\eta \leqslant \zeta \Rightarrow \eta_t \leqslant \zeta_t$, a.s., and translation invariance

$$\tilde{\mathbb{E}}^{(\eta,\zeta)} \xi_t(y) = \tilde{\mathbb{E}}^{(\eta,\zeta)} \xi_t(0)$$

$$\sum_y p(y, x) = \sum_y p(0, x-y) = 1$$

Hence, we arrive at

$$\tilde{\mathbb{E}}^{(\eta,\zeta)} F(\xi_t(x)) \leqslant \tilde{E}^{(\eta,\zeta)}(F(\xi_1(x)) \exp[-\varepsilon(t-1)] \qquad t \geqslant 1, \quad x \in S$$

(c) Let $\Lambda_N = [-N+1, N]^d \subset \mathbb{Z}^d$ and regard Λ_N as the torus $S_N = \mathbb{Z}^d/(2N\mathbb{Z}^d)$, the factor group. On S_N, we can introduce a shift operator in a natural way and translation invariance is meaningful. Next, for a given translation-invariant transition probability $p(x, y)$ on $\mathbb{Z}^d$, we can introduce $p_N(x, y)$ on S_N with the same property:

$$p_N(0, x) = p(0, x) \Big/ \sum_{z \in S_N} p(0, z), \qquad x \in S_N$$

Here I have identified $x \in S_N$ as an element in $\mathbb{Z}^d$.

Applying (a) and (b) to the present case with an obvious change of notations, we get

$$\tilde{\mathbb{E}}_N^{(\eta,\zeta)} F(\xi_t(0)) \leqslant \tilde{\mathbb{E}}_N^{(\eta,\zeta)} F(\xi_1(0)) \exp[-\varepsilon(t-1)], \qquad t \geqslant 1 \tag{2.1}$$

for any initial $\eta \leqslant \zeta$, η, $\zeta \in E_N^s$.

(d) Now, let us go back to Λ_N. Regard the above process $\tilde{P}_N(t)$ as a process on $\mathbb{Z}_+^{\Lambda_N}$. Let $\tilde{\Omega}$ denote the infinite-dimensional coupling operator constructed in the same way as in (a). It is easy to check that for every $x \in \mathbb{Z}^d$, if we put $h_x(\eta, \zeta) = \eta(x) + \zeta(x)$, then

$$\tilde{\Omega} h_x(\eta, \zeta) \leqslant c[1 + h_x(\eta, \zeta)]$$

for some constant $c < \infty$. Moreover, the interaction between two boxes [i.e., $\eta(x)\, p(x, y)$] is at most linear. These two facts enable us to find a Markov process with generator $\tilde{\Omega}$.[4] An alternative way is to take a weak limit $\tilde{\mathbb{P}}$ in the usual Skorohod topology, which is a solution to the Martingale problem for the operator $\tilde{\Omega}$ (see ref. 9 for details). Thus, from Lemma 2.2 and (2.1), it follows that

$$\tilde{\mathbb{E}}^{(\eta,\zeta)} F(\xi_t(0)) \leqslant \mathbb{E}^{(\eta,\zeta)} F(\xi_1(0)) \exp[-\varepsilon(t-1)] \tag{2.2}$$

Since the original process is unique, it does not depend on the ways of different finite-dimensional approximations. Hence each marginal distribution of $\tilde{\mathbb{P}}$ coincides with the original process. In particular, we have

$$\tilde{\mathbb{E}}^{(\eta,\zeta)} F(\xi_1(0)) \leqslant \mathbb{E}^{\zeta} \zeta_1(0) - \mathbb{E}^{\eta} \eta_1(0) < \infty \tag{2.3}$$

(e) Finally, let η_t^n denote the Markov process starting from $\eta(x) = n$, $x \in \mathbb{Z}^d$. By the attractiveness of the original process, we have $\zeta_t^n \leqslant \zeta_t^{n+1}$, a.s. Following Ding *et al.*,[7] we can construct a common probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{E}})$ on which the process (η_t, ζ_t) lives and

$$\zeta_t^n \uparrow \zeta_t^\infty, \qquad \bar{\mathbb{P}}\text{-a.s.}$$

By Lemma 2.2,

$$\bar{\mathbb{E}} \zeta_1^\infty(0) < \infty \tag{2.4}$$

By (2.2)–(2.4) and Fatou's lemma, we get

$$\begin{aligned} &\bar{\mathbb{E}} F(\zeta_t^\infty(0) - \eta_t^0(0)) \\ &\quad \leqslant \varliminf_{n \to \infty} \bar{\mathbb{E}} F(\zeta_1^n(0) - \eta_1^0(0)) \exp[-\varepsilon(t-1)] \\ &\quad \leqslant \bar{\mathbb{E}}(\zeta_1^\infty(0) - \eta_1^0(0)) \exp[-\varepsilon(t-1)], \qquad t \geqslant 1 \end{aligned}$$

Using Fatou's lemma again, we finally obtain

$$\begin{aligned}
&\tilde{\mathbb{E}}F(\zeta_\infty^\infty(0)-\eta_\infty^0(0)) \\
&\quad \leqslant \varliminf_{t\to\infty} \tilde{\mathbb{E}}F(\zeta_t^\infty(0)-\eta_t^0(0)) \\
&\quad \leqslant \varliminf_{t\to\infty} \tilde{\mathbb{E}}F(\zeta_1^\infty(0)-\eta_1^0(0))\exp[-\varepsilon(t-1)]=0
\end{aligned}$$

Since $u_l>0$ for all $l\geqslant 0$, this proves that

$$\zeta_\infty^\infty(0)\overset{\mathrm{d}}{=}\eta_\infty^0(0) \qquad \text{QED}$$

Remark 2.1. In view of (b) in the previous proof, what we need is to choose a sequence $\{v_l\colon l\geqslant 0\}$ such that $0<v_l\leqslant u_l$ for all $l\geqslant 0$. But it is easy to see that the largest choice of $\{v_l\}$ is just $\{u_l\}$. One may ask whether one can improve the theorem by setting $\varepsilon=0$ in the formulation of $u_l=u_l(\varepsilon)$. The answer is negative. Consider the linear growth model with $\beta_1=\delta_1$; then $u_l(0)=1$ for all $l\geqslant 0$. But the translation-invariant stationary distributions are not unique.[8] This is the main difference between the finite-dimensional case and the infinite-dimensional one. In other words, we can take $\varepsilon=0$ in the former case by studying the successful couplings, but we cannot do so for the latter case.

On the other hand, the estimate

$$\tilde{\mathbb{E}}^{(\eta,\zeta)}\sum_y \xi_t(y)\,p(y,x)u_{\xi(u)}\leqslant \tilde{\mathbb{E}}^{(\eta,\zeta)}\sum_y \xi_t(y)\,p(y,x)$$

used in (b) may not be sharp, but I have no way to improve it at the moment.

Remark 2.2. By Lemma 2.1, the sequence $\{u_l\}$ is decreasing, hence $\rho(k,l)=F(|k-l|)$ is a metric on $\mathbb{Z}_+$. Furthermore,

$$p(\eta,\zeta)=\sum_x \alpha(x)\,\rho(\eta(x),\zeta(x))$$

defines a metric on E_0, and so we have a Kantorovich distance

$$K(P,Q)=\inf_{\tilde{P}}\int p(\eta,\zeta)\,\tilde{P}(d\eta,d\zeta)$$

where P and Q are probabilities on E_0 and $\tilde{P}$ varies over all coupling probabilities of P and Q. In this notation, we have indeed proved that

$$
\begin{aligned}
K(P(t, \eta, \cdot), P(t, \zeta, \cdot)) \\
&\leqslant \tilde{\mathbb{E}}^{(\eta, \zeta)} p(\eta_t, \zeta_t) \\
&\leqslant \sum_x \alpha(x)\, \mathbb{E}F(\zeta_t^\infty(0) - \eta_t^0(0)) \\
&\leqslant \left[\sum_x \alpha(x)\right] \exp[-\varepsilon(t-1)]\, \mathbb{E}(\zeta_1^\infty(0) - \eta_1^0(0)) \\
&\to 0 \qquad \text{as} \quad t \to \infty
\end{aligned}
$$

Remark 2.3. Having the above probability distance $K(P, Q)$ in mind, along the lines of refs. 3 and 5, we can prove Theorem 1.1, even more simply for the more general case where the degree of the death rate may be equal to one and $p(x, y)$ is not necessarily translation invariant. Of course, the condition

$$\sup_y \sum_x p(x, y) < \infty$$

is needed. The reason I do not adopt the simpler proof here is that I want to cover some other cases, where some sequence $\{u_l'\}$ does not define a metric on $\mathbb{Z}_+$.

3. DISCUSSION OF OTHER CRITERIA

The coupling for the birth–death processes (i.e., the reaction part) used in the last section simply makes the two marginals jump to places that are as close as possible. Of course, one can use other couplings. Here are some examples.

Fix $x \in \mathbb{Z}^d$ and let $\eta(x) = k$ and $\xi(x) = \zeta(x) - \eta(x) = l \geqslant 0$. If $l = 0$, always make the marginals move simultaneously (in the box x).

(i) *Basic coupling* $\tilde{\Omega}_b$. When $l = 2$, choose

$$
\begin{aligned}
(k, k+l) &\to (k+1, k+1) \quad \text{at rate } b(k) \wedge a(k+l) \\
&\to (k-1, k+l) \quad \text{at rate } a(k) \\
&\to (k, k+l+1) \quad \text{at rate } b(k+l)
\end{aligned}
$$

I repeat here that I omit the two rates $(A-B)^+$ and $(B-A)^+$ whenever there is a rate $A \wedge B$ (say). When $l = 1$ or $\geqslant 3$ one simply makes the two marginals move independently.

(ii) *Classical coupling* $\tilde{\Omega}_c$. For $l \geqslant 1$, the marginals move independently.

(iii) *March coupling* $\tilde{\Omega}_m$. For $l \geqslant 1$,

$$
\begin{aligned}
(k, k+l) &\to (k+1, k+l+1) \qquad \text{at rate } b(k) \wedge b(k+l) \\
&\to (k-1, k+l-1) \qquad \text{at rate } a(k) \wedge a(k+l)
\end{aligned}
$$

This coupling has been used often in the study of reaction-diffusion processes. It has the advantage of being easy to handle and, as mentioned before, is good enough for the linear growth model.

Combining the above couplings, one may produce other couplings. For example, by combining (ii) with (iii), we get the following coupling.

(iv) *Coupling* $\tilde{\Omega}_{cm}$. If $l \leqslant 1$, use $\tilde{\Omega}_c$; if $l \geqslant 2$, use $\tilde{\Omega}_m$.

For all of these couplings (recall that the coupling for diffusion part is kept fixed), there are corresponding criteria for the ergodicity of reaction-diffusion processes. To state them, set

$$
\begin{aligned}
&u_0^b = u_0^c = u_0^{cm} = u_0^m = 1 \\
&u_1^b = u_1^c = u_1^{cm} = u_1 = u_1(\varepsilon) \\
&u_2^b = u_2 = u_2(\varepsilon) \\
&u_l^c = u_l^c(\varepsilon) \\
&= \left(\inf_{k \geqslant 0} \left\{ [b(k) + a(k+l) + l] u_{l-1}^c - l - \varepsilon \sum_{j=0}^{l-1} u_j^c \right\} \right. \\
&\qquad \left. \times [a(k) + b(k+l) + \varepsilon]^{-1} \right) \vee 0, \qquad l \geqslant 2
\end{aligned}
\tag{3.1}
$$

Replacing u_{l-1}^c with u_{l-1}^b in the right-hand side of (3.1), one defines $u_l^b = u_l^b(\varepsilon)$ for $l \geqslant 3$. Next, set

$$
\begin{aligned}
&u_l^m = u_l^m(\varepsilon) \\
&= \left[\left(\inf_{k \geqslant 0} \left\{ [a(k+l) - a(k) + i] u_{l-1}^m - l - \varepsilon \sum_{j=0}^{l-1} u_j^m \right\} \right. \right. \\
&\qquad \left. \left. \times [b(k+l) - b(k) + \varepsilon]^{-1} \right) \vee 0 \right] \wedge 1, \qquad l \geqslant 1
\end{aligned}
\tag{3.2}
$$

Similarly, replacing u_{l-1}^m with u_{l-1}^{cm} in the right-hand side of (3.2), one defines $u_l^{cm} = u_l^{cm}(\varepsilon)$ for $l \geqslant 2$.

Among all of these sequences, (u_l^c), (u_l^m), and (u_l^{cm}) are easier for computation. Following the proof given in the last section, we obtain the following result.

Proposition 3.1. Under (H), for each $r = b, c, m$, or cm, if $u_l^r(\varepsilon) > 0$ for some $\varepsilon > 0$ and all $l \geqslant 0$, then the process is ergodic.

I refer to the criterion given by Proposition 3.1 as the u^r-criterion ($r = b, c, m, cm$). Now I compare these criteria with the original result.

Proposition 3.2. Under (H), if (1.2) holds, then there exists an $\varepsilon > 0$ such that

$$u_l(\varepsilon) = u_l^r(\varepsilon) = 1$$

for all $l \geqslant 0$ and $r = b, c, m, cm$.

Proof. Since the proof is the same for different cases, we check only that

$$u_l^m(\varepsilon) = 1, \qquad l \geqslant 0$$

It suffices to show that

$$[\Delta_l a(k) - l\varepsilon]/[\Delta_l b(k) + \varepsilon] \geqslant 1, \qquad k \geqslant 0, \quad l \geqslant 1$$

where $\Delta_l a(k) = a(k+l) - a(k)$ and $\Delta_l b(k) = b(k+l) - n(k)$. This is nothing but the condition (1.2). QED

In order to compare the above criteria with the u-criterion, I first make some remarks. First, we can assume that $b(k) \neq \text{const}$, since for this special case, the condition (1.2) is already good enough. Second, it seems hard to handle the general case, and so I restrict myself to the extreme case $\varepsilon = 0$. This is meaningful because if we replace $-l - \varepsilon \sum_{j=0}^{l-1} u_j^r$ with $-(1+\varepsilon)l$ in (3.1) and (3.2), the proof given below still works well. Even the extreme case is stronger than in the original case, but it is the most practical one (see next section). On the other hand, since the coupling for the diffusion part is fixed, we are indeed comparing the couplings of one-dimensional birth–death processes. Hence ε can be allowed to be zero, at least in the study of successful coupling.[6]

Let

$$\bar{u}_l = u_l(0), \qquad \bar{u}_l^r = u_l^r(0), \qquad r = b, c, m, cm, \quad l \geqslant 0 \tag{3.3}$$

For the successful coupling, one requires that $\sum \bar{u}_l = \infty$ ($\sum \bar{u}_l^r = \infty$).

The following trivial fact will be used in the next section.

Lemma 3.3. For any $\varepsilon > 0$ and $l \geqslant 0$, we have

$$\bar{u}_l \geqslant u_l(\varepsilon), \qquad \bar{u}_l^r \geqslant u_l^r(\varepsilon), \qquad r = b, c, m, cm$$

Proposition 3.4. Under (H), if $b(k)$ is not a constant, then for every $l \geqslant 0$, we have

$$\bar{u}_l \geqslant \bar{u}_l^b \geqslant \bar{u}_l^c \qquad \text{and} \qquad \bar{u}_l^{cm} \geqslant \bar{u}_l^m$$

As for the gap between $\bar{u}_l^c$ and $\bar{u}_l^{cm}$, we have the following conclusions:

(i) If there exists an $l_0 \geqslant 1$ such that $\bar{u}_{l_0}^c \geqslant \bar{u}_{l_0}^{cm}$ and $\{\bar{u}_l^c\}_{l \geqslant l_0}$ is strictly decreasing, then so is $\{\bar{u}_l^{cm}\}_{l \geqslant l_0}$ and $\bar{u}_l^c \geqslant \bar{u}_l^{cm}$ for all $l \geqslant l_0$.

(ii) If there exists an $l_0 \geqslant 0$ such that $\bar{u}_{l_0}^c \geqslant \bar{u}_{l_0}^{cm}$ and $\bar{u}_l^c = \bar{u}_{l_0}^c$ for all $l \geqslant l_0$, then $\sum_0^\infty \bar{u}_l^{cm} \leqslant \sum_0^\infty \bar{u}_l^c = \infty$.

In the above cases, we certainly have $\bar{u}_l^{cm} > 0$ $(\forall l \geqslant 0) \Rightarrow \bar{u}_l^c > 0$ $(\forall l \geqslant 0)$. Moreover, if the degree of the death rates equals two or three, then either (i) or (ii) will happen. In general:

(iii) If we replace $\bar{u}_l^{cm}$ by $\tilde{u}_l^{cm} = \bar{u}_l^{cm} \wedge \bar{u}_{l-1}^{cm}$ $(l \geqslant 1)$, then $\bar{u}_l^c \geqslant \tilde{u}_l^{cm}$ for all $l \geqslant 0$.

In the above sense, the coupling used for the u-criterion is the best one.

Proof. (a) As in the proof of Lemma 2.1, $(\bar{u}_l^b)$ and $(\bar{u}_l^c)$ are decreasing and bounded by 1 from above. Hence, it should be clear that $\bar{u}_l \geqslant \bar{u}_l^b \geqslant \bar{u}_l^c$ for all $l \geqslant 0$.

(b) For $\bar{u}_l^{cm} \geqslant \bar{u}_l^m$, $l \geqslant 0$, we need only to check that $\bar{u}_1^{cm} \geqslant \bar{u}_1^m$. Recall that $\Delta a(k) = \Delta_1 a(k) = a(k+1) - a(k)$ and so on. We have

$$\begin{aligned}
\bar{u}_1^m &\Leftrightarrow \Delta a(k) \geqslant \Delta b(k), \qquad \forall k \geqslant 0 \\
&\Leftrightarrow b(k) + a(k+1) \geqslant a(k) + b(k+1), \qquad \forall k \geqslant 0 \\
&\Leftrightarrow \bar{u}_1^{cm} = 1
\end{aligned}$$

Thus, we may assume that

$$\mathscr{K}_1 = \{k \geqslant 0 \colon \Delta a(k) - \Delta b(k) < 0\} \neq \varnothing$$

which is a finite subset of $\mathbb{Z}_+$. Then

$$\begin{aligned}
1 > \bar{u}_1^{cm} &= \min_{k \geqslant 0} [b(k) + a(k+1)]/[a(k) + b(k+1)] \\
&= \min_{k \in \mathscr{K}_1} \left(\frac{\Delta a(k) - \Delta b(k)}{a(k) + b(k+1)} + 1 \right) \\
&> \min_{k \in \mathscr{K}_1} \left(\frac{\Delta a(k) - \Delta b(k)}{\Delta b(k)} + 1 \right) \geqslant \bar{u}_1^m
\end{aligned}$$

For the strict inequality I have used $a(k) + b(k+1) > b(k) > 0$.

(c) In the case (i), we have

$$\mathscr{K}_l = \left\{ k \geqslant 0: \frac{[b(k) + a(k+l) + l]\bar{u}_{l-1}^c - l}{a(k) + b(k+l)} < \bar{u}_{l-1}^c \right\}$$
$$= \{k \geqslant 0: [\Delta_l a(k) - \Delta_l b(k) + l]\bar{u}_{l-1}^c - l < 0\}$$
$$\neq \varnothing, \qquad l \geqslant l_0 + 1$$

By assumption, $\bar{u}_{l_0}^c \geqslant \bar{u}_{l_0}^{cm}$. Now, suppose that this holds for some $l \geqslant l_0$; then

$$1 > \bar{u}_l^c = \left\{ \min_{k \in \mathscr{K}_l} \frac{[\Delta_l a(k) - \Delta_l b(k) + l]\bar{u}_{l-1}^c - l}{a(k) + b(k+l)} + \bar{u}_{l-1}^c \right\} \vee 0$$
$$\geqslant \left\{ \min_{k \in \mathscr{K}_l} \frac{[\Delta_l a(k) - \Delta_l b(k) + l]\bar{u}_{l-1}^c - l}{\Delta_l b(k)} + \bar{u}_{l-1}^c \right\} \vee 0$$
$$= \left\{ \min_{k \in \mathscr{K}_l} \frac{[\Delta_l a(k) + l]\bar{u}_{l-1}^c - l}{\Delta_l b(k)} \right\} \vee 0$$
$$\geqslant \left\{ \min_{k \in \mathscr{K}_l} \frac{[\Delta_l a(k) + l]\,\bar{u}_{l-1}^{cm} - l}{\Delta_l b(k)} \right\} \vee 0$$
$$\geqslant \bar{u}_l^{cm}$$

By induction, we have $\bar{u}_l^c \geqslant \bar{u}_l^{cm}$ for all $l \geqslant l_0$.

Next, suppose that $\bar{u}_l^{cm} \geqslant \bar{u}_{l-1}^{cm}$ for some $l \geqslant l_0 + 1$. Then by definition we would not only have that

$$\min_{k \geqslant 0} [\Delta_l a(k) - \Delta_l b(k)]/l + 1 > 0 \tag{3.4}$$

but also that

$$\bar{u}_{l-1}^{cm} \geqslant \max_{k \geqslant 0} \{[\Delta_l a(k) - \Delta_l b(k)]l + 1\}^{-1} \tag{3.5}$$

Since we know from (c) that $\bar{u}_{l-1}^c \geqslant \bar{u}_{l-1}^{cm}$, replacing $\bar{u}_{l-1}^{cm}$ by $\bar{u}_{l-1}^c$, (3.4) and (3.5) would still hold. This implies that $\bar{u}_{l+1}^c = \bar{u}_l^c$, which contradicts our assumption. We have thus complete the proof of case (i) as well as that for case (iii).

(d) Consider case (ii). The special situation that $l_0 = 0$ was treated in Proposition 3.2. Hence, we assume that $l_0 \geqslant 1$. If for all $l \geqslant l_0$, $\bar{u}_l^{cm} \leqslant \bar{u}_l^c$, then the assertion is trivial. Assume that there exists some $l \geqslant l_0 + 1$ for which $\bar{u}_l^{cm} > \bar{u}_{l_0}^c$. Let l_1 be the first one after l_0 having this property. The condition $\bar{u}_{l_0}^c = \bar{u}_{l_0+1}^c = \cdots$ simply means that

$$\inf_{l \geqslant l_0 + 1} \{\min_{k \geqslant 0} [\Delta_l a(k) - \Delta_l b(k)]/l + 1\} = r > 0$$

and that $\bar{u}_{l_0}^c \geqslant r^{-1}$. Thus, if $\bar{u}_{l_1}^{cm} > \bar{u}_{l_0}^c$, if follows from the last paragraph that we should have $\bar{u}_{l+1}^{cm} \geqslant \bar{u}_l^{cm}$ for all $l \geqslant l_1$. Hence $\sum \bar{u}_l^c = \sum \bar{u}_l^{cm} = \infty$.

(e) Finally, consider the situation where the degree of the death rate equals two or three. These contain the first and the second Schlögl models. In both situations, if $\{\bar{u}_l^c\}_{l \geqslant 0}$ is strictly decreasing, then we just have case (i). Otherwise, let l_0 be the first integer so that

$$\bar{u}_0^c > \bar{u}_1^c > \cdots > \bar{u}_{l_0}^c \qquad \text{and} \qquad \bar{u}_{l_0+1}^c = \bar{u}_{l_0}^c$$

Then the proof in (c) indicates that

$$\bar{u}_0^{cm} > \bar{u}_1^{cm} > \cdots \bar{u}_{l_0}^{cm}, \qquad \bar{u}_l^c \geqslant \bar{u}_l^{cm} \qquad \text{for} \quad l \leqslant l_0 \tag{3.6}$$

Now, I claim that

$$\bar{u}_l^c = \bar{u}_{l_0}^c \qquad \text{for all} \quad l \geqslant l_0 \tag{3.7}$$

If this holds, then (3.6) and (3.7) imply that we are in case (ii). To prove (3.7), recall that for $l \geqslant 1$, $\bar{u}_l^c = \bar{u}_{l-1}^c$ if and only if (3.4) and (3.5) hold with $\bar{u}_{l-1}^{cm}$ replaced by $\bar{u}_{l-1}^c$. These modified conditions are denoted by (3.4′) and (3.5′) respectively. Let us now study the case of degree $m = 1$ or 2, respectively.

For $m = 1$, we have

$$\min_{k \geqslant 0} [\Delta_l a(k) - \Delta_l b(k)]/l + 1 = \delta_1 - \beta_1 + 1 + \delta_2(l - 1)$$

which is increasing in l. Hence, (3.4′) and (3.5′) are satisfied for all $l \geqslant l_0 + 1$.

For $m = 2$, the situation is more complicated. In that case,

$$\begin{aligned} h(k, l) &= [\Delta_l a(k) - \Delta_l b(k)]/l + 1 \\ &= (\delta_1 - \beta_1) + (\delta_2 - \beta_2)(l + 2k - 1) + \delta_3[l^2 + 3(k-1)l \\ &\quad + 3k^2 - 6k + 2] + 1 \\ &= 3\delta_3 \left[k + \left(\frac{l}{2} - 1 + \frac{\delta_2 - \beta_2}{3\delta_3} \right) \right]^2 - 3\delta_3 \left(\frac{l}{2} - 1 + \frac{\delta_2 - \beta_2}{3\delta_3} \right)^2 \\ &\quad + (\delta_1 - \beta_1) + (\delta_2 - \beta_2)(l - 1) + \delta_3(l - 1)(l - 2) + 1 \end{aligned}$$

For fixed $l \geqslant 1$, if $l/2 - 1 + (\delta_2 - \beta_2)/3\delta_3 \equiv \alpha > 0$, then $h(\cdot, l)$ achieves its minimum at $k = 0$ and

$$\begin{aligned} h(l) &= \min_{k \geqslant 0} h(k, l) \\ &= \delta_1 - \beta_1 + (\delta_2 - \beta_2)(l - 1) + \delta_3(l - 1)(l - 2) + 1 \end{aligned}$$

It is easy to check (since $\alpha > 0$) that in the present situation, $h(l)$ is increasing in l for $l > 2[1-(\delta_2-\beta_2)/3\delta_3]$. If $\alpha < 0$, then $h(\cdot, l)$ achieves its minimum at $k_0 = k_0(l)$, the nonnegative integer closest to $-\alpha$. Moreover,

$$\begin{aligned} h(l) &= \min_{k \geqslant 0} h(k, l) \\ &= 3\delta_3\left(k_0+\frac{l}{2}-1+\frac{\delta_2-\beta_2}{3\delta_3}\right)^2 - 3\delta_3\left(\frac{l}{2}-1+\frac{\delta_2-\beta_2}{3\delta_3}\right)^2 \\ &\quad + (\delta_1-\beta_1)+(\delta_2-\beta_2)(l-1)+\delta_3(l-1)(l-2)+1 \\ &= 3\delta_3\left(k_0+\frac{l}{2}-1+\frac{\delta_2-\beta_2}{3\delta_3}\right)^2 + \frac{1}{4}\delta_3 l^2 - \delta_3 \\ &\quad + (\delta_1-\beta_1)+1+(\delta_2-\beta_2)-\frac{(\delta_2-\beta_2)^2}{3\delta_3} \end{aligned} \tag{3.8}$$

Since the first term on the right-hand side of (3.8) is less than $\frac{3}{4}\delta_3$, it follows that the function $h(l)$ is increasing in l not only on the interval $1 \leqslant l < 2[1-(\delta_2-\beta_2)/3\delta_3]$, but also for all $l \geqslant 1$. This again shows that the above two conditions (3.4′) and (3.5′) are satisfied. QED

Remark 3.1. The idea in Remarks 2.2 and 2.3 can also be applied to the (u_l^b)- and (u_l^c)-sequences but not to the (u_l^{cm})- and (u_l^m)-sequences. The reason is that, for instance,

$$\sum_{0 \leqslant j \leqslant |k-l|-1} u_j^{cm}$$

is in general no longer a metric on $\mathbb{Z}_+$.

4. APPLICATIONS

In this section, I finally prove that the systems are ergodic if β_0 is sufficient large (Theorem 4.8). But first, let us study Schlögl's first and second models more carefully. For the first one, we have

$$\Delta_l a(k) = l[\delta_1+\delta_2(l+2k-1)], \qquad \Delta_l b(k) = \beta_1 l$$

and for the second one,

$$\begin{aligned} &\Delta_l a(k) = l\{\delta_1+\delta_3[l^2+3(k-1)l+3k^2-6k+2]\} \\ &\Delta_l b(k) = l\beta_2(l+2k-1) \end{aligned}$$

Lemma 4.1. For the first Schlögl model, we have

$$u_1(\varepsilon)=\begin{cases}1 & \text{if} \quad \delta_1>\beta_1\\ \dfrac{\delta_1+\beta_0-\varepsilon}{\beta_1+\beta_0+\varepsilon} & \text{if} \quad \delta_1\leqslant\beta_1, \quad \varepsilon \text{ small enough}\end{cases}$$

For the second one, we have

$$u_1(\varepsilon)=\begin{cases}1 \quad \text{if} \quad (3\delta_3+2\beta_2)^2<12\delta_1\delta_3\\ \min\limits_{k\in\mathscr{K}_0}\dfrac{b(k)+a(k+1)-\varepsilon}{a(k)+b(k+1)+\varepsilon} \qquad \text{if} \quad (3\delta_3+2\beta_2)^2\geqslant 12\delta_1\delta_3\\ \qquad\qquad\qquad\qquad \text{and } \varepsilon \text{ small enough}\end{cases}$$

where

$$\mathscr{K}_0=\{k\geqslant 0: |6\delta_3 k-(3\delta_3+2\beta_2)|\leqslant[(3\delta_3+2\beta_2)^2-12\delta_1\delta_3]^{1/2}\}$$

In particular, if $\delta_1\leqslant 2\beta_2$, then $u_1(\varepsilon)\leqslant(2\delta_1+\beta_0-\varepsilon)/(\delta_1+2\beta_2+\beta_0+\varepsilon)$ for small enough $\varepsilon>0$.

From Lemma 3.3, we easily get the following results.

Proposition 4.2. In order for the u^{cm}-criterion to be applicable, it is necessary that

$$\bar{u}_l^{cm}>\max_{k\geqslant 0}\{\Delta_l a(k)/l+1\}^{-1}=\{a(l)/l+1\}^{-1}, \qquad l\geqslant 1$$

For the first and second Schlögl models, this becomes

$$\bar{u}_l^{cm}>\{\delta_1+1+\delta_2 l\}^{-1}, \qquad l\geqslant 1$$

and

$$\bar{u}_l^{cm}>\{1+\delta_1+\delta_3 l(l-1)\}^{-1}, \qquad l\geqslant 1$$

respectively. In particular, for the first model, if $\beta_1>\delta_1$, then

$$\delta_1+(\delta_1+\delta_2)(\delta_1+\beta_0)>\beta_1$$

is a necessary condition; for the second model, if $\delta_1 \leqslant 2\beta_2$, then

$$\delta_1(1+2\delta_1+\beta_0) > 2\beta_2$$

is necessary.

Proposition 4.3. In order for the u-criterion to be applicable, it is necessary that

$$\bar{u}_{l-2} > \max_{k \geqslant 0}\{[b(k)+a(k+l)]/l+1\}^{-1}$$
$$= \{[\beta_0+a(l)]/l+1\}^{-1}, \qquad l \geqslant 2$$

For the Schlögl models, this becomes

$$\bar{u}_{l-2} > \{\beta_0/l+\delta_1+\delta_2(l-1)+1\}^{-1}$$

and

$$\bar{u}_{l-2} > \{\beta_0/l+\delta_1+\delta_3(l-1)(l-2)+1\}^{-1}$$

respectively. In particular, for the first model, if $\beta_1 > \delta_1$, then

$$\delta_1(\delta_1+2\delta_2)+\beta_0(\beta_0+4\delta_1+6\delta_2)/3 > \beta_1$$

is a necessary condition; for the second model, if $\delta_1 \leqslant 2\beta_2$, then

$$\delta_1+(\delta_1+2\delta_3+\beta_0/3)(\beta_0+2\delta_1) > 2\beta_2$$

is necessary.

To obtain some precise ergodic conditions by using the u-criterion (u'-criterion), one should study both the upper and the lower bounds of the u_l. For simplicity, I consider only the latter lower one. I first introduce more practical versions of the present criteria. Put

$$\tilde{u}_0 = u_0, \qquad \tilde{u}_1 = \tilde{u}_1(\varepsilon) = u_1(\varepsilon)$$
$$\tilde{u}_l = \tilde{u}_l(\varepsilon) = (\inf_{k \geqslant 0}\{[b(k) \vee a(k+l)+l-\varepsilon]\tilde{u}_{l-1}$$
$$+[b(k) \wedge a(k+l)-\varepsilon]\tilde{u}_{l-2}-l-\varepsilon(l-2)\}$$
$$\times [a(k)+b(k+l)+\varepsilon]^{-1}) \vee 0, \qquad l \geqslant 2$$

Clearly,

$$\tilde{u}_l \leqslant u_l, \qquad l \geqslant 0$$

The criterion corresponding to the $(\tilde{u}_l)$-sequence is called the $\tilde{u}$-criterion for now.

For the $\tilde{u}$-criterion, we have a comparison theorem for two processes with different rates (b^i, a^i), $i=1, 2$. Here, I consider only the following cases:

(i) $b^1(k)=b^2(k)=b(k)$ and $a^1(k)\geqslant a^2(k)$, $k\geqslant 0$.

(ii) $a^1(k)=a^2(k)=a(k)$, $b^1(k)\geqslant b^2(k)$, $k\geqslant 0$, but only the pure birth rates of $b^1(k)$ and $b^2(k)$ can be different.

In both cases, I use the same notation $(\tilde{u}_l^i)$ for the sequence defined above corresponding to the rates (b^i, a^i), $i=1, 2$.

Proposition 4.4. For the above two cases (i) and (ii), we have

$$\tilde{u}_l^1 \geqslant \tilde{u}_l^2 \qquad \text{for all} \quad l\geqslant 0$$

The proof is very much similar to that of Proposition 3.4 and hence is omitted here. Note that the above two cases are not symmetric. This means that a special role is played by the pure birth rate β_0.

Similarly, one can introduce the corresponding versions of (u_l^r) and their criteria. However, in what follows I will not mention the $\tilde{u}^r$-criteria any further. More precisely, I look for estimates of the type $\tilde{u}_l(\varepsilon)\geqslant \gamma(1+\alpha l)^{-1}$ for some $\varepsilon>0$ and all $l\geqslant 1$. Then, one certainly has $u_l(\varepsilon)\geqslant \gamma(1+\alpha l)^{-1}$ for the same $\varepsilon>0$ and all $l\geqslant 1$. Moreover, I only discuss the first Schlögl model.

For the u^{cm}-criterion, one has the following result.

Corollary 4.5. For the first Schlögl model, if

$$\delta_1\left\{1+\delta_2\frac{(1+\beta_0/\delta_1)[1+2(\delta_1+\delta_2)]}{(1+\delta_1+\delta_2)(1+\delta_1+\beta_0)+\delta_2}\right\}>\beta_1 \tag{4.1}$$

then

$$u_l^{cm}(\varepsilon)\geqslant \gamma(1+\alpha l)^{-1} \qquad \text{for some } \varepsilon>0 \text{ and all } l\geqslant 1 \tag{4.2}$$

where $\gamma=1$ and $\alpha=0$ if $\delta_1>\beta_1$; otherwise, $\alpha=\delta_2/(1+\delta_1)$ and

$$\frac{1}{1+\delta_1}\left[1+\frac{1}{1+\delta_1+2\delta_2-\beta_1}\right]<\gamma<\frac{(1+\delta_1+\delta_2)(\delta_1+\beta_0)}{(1+\delta_1)(\beta_1+\beta_0)}$$

In particular, (4.2) holds for the specific case $\gamma=1$ if

$$\delta_1\left[1+\delta_2\frac{1+1\wedge(\beta_0/\delta_1)}{1+\delta_1}\right]>\beta_1 \tag{4.3}$$

The proof is omitted here since it is similar to the next one. Indeed, we have

$$u_l^{cm} = \left[\left(\left\{ [\delta_1 + \delta_2(l-1) + 1] u_{l-1}^{cm} - 1 - \varepsilon \sum_{j=0}^{l-1} u_j^{cm}/l \right\} \times (\beta_1 + \varepsilon/l)^{-1} \right) \vee 0 \right] \wedge 1, \qquad l \geqslant 2$$

which does not involve the variable k.

Remark 4.1. Comparing the new sufficient condition (4.3) with (1.3), we see that the new one is much better. Next, in view of Proposition 4.2, the estimate of (u_l^{cm}) given here is nearly necessary. Finally, for large enough β_0, (4.1) reduces to

$$(\delta_1 + \delta_2)\left(1 + \frac{\delta_2}{1 + \delta_1 + \delta_2}\right) > \beta_1 \tag{4.4}$$

To study the u-criterion, we need the following simple observation.

Lemma 4.6. For the first Schlögl model, we have

$$u_l = u_{l-1} \quad \text{if} \quad [a(l) + l - \beta_1 l - \varepsilon] u_{l-1} + [\beta_0 \wedge a(l)](u_{l-2} - u_{l-1}) \geqslant l + \varepsilon \sum_{j=0}^{l-1} u_j$$

$$u_l = \left(\left\{ [\beta_0 \vee a(l) + l] u_{l-1} + [\beta_0 \wedge a(l)] u_{l-2} - \varepsilon - \varepsilon \sum_{j=0}^{l-1} u_j \right\} \times [b(l) + \varepsilon]^{-1} \right) \vee 0, \qquad l \geqslant 2, \qquad \text{otherwise}$$

Proof. For every $l \geqslant 2$, $u_l = u_{l-1}$ if and only if

$$[b(k) \vee a(k+l) + l] u_{l-1} + [b(k) \wedge a(k+l)] u_{l-2} - [a(k) + b(k+l) + \varepsilon] u_{l-1} \geqslant l + \varepsilon \sum_{j=0}^{l-1} u_j, \qquad \forall k \geqslant 0$$

That is,

$$[\Delta_l a(k) - \Delta_l b(k) + l - \varepsilon] u_{l-1} + [b(k) \wedge a(k+l)](u_{l-2} - u_{l-1}) \geqslant l + \varepsilon \sum_{j=0}^{l-1} u_j, \qquad \forall k \geqslant 0 \tag{4.5}$$

Since the left-hand side of (4.5) is increasing in k, it follows that (4.5) is equivalent to

$$[a(l)+l-\beta_1 l-\varepsilon]u_{l-1}+[\beta_0 \wedge a(l)](u_{l-2}-u_{l-1}) \geqslant l+\varepsilon \sum_{j=0}^{l-1} u_j$$

This proves the first part of the lemma. Conversely, if (4.5) does not hold, then we have

$$\begin{aligned}
&\left\{[b(k) \vee a(k+l)+l]u_{l-1}+[b(k) \wedge a(k+l)]u_{l-2}\right.\\
&\qquad \left.-l-\varepsilon \sum_{j=0}^{l-1} u_j\right\} \Big/ [a(k)+b(k+l)+\varepsilon]\\
&\quad =\left\{[\Delta_l a(k)-\Delta_l b(k)+l-\varepsilon]u_{l-1}\right.\\
&\qquad \left.+[b(k) \wedge a(k+l)](u_{l-1}-u_{l-2})-l-\varepsilon \sum_{j=0}^{l-1} u_j\right\}\\
&\qquad \times [a(k)+b(k+l)+\varepsilon]^{-1}+u_{l-1}\\
&\quad \geqslant \left\{[a(l)+l-\beta_1 l-\varepsilon]u_{l-1}+[\beta_0 \wedge a(l)](u_{l-2}-u_{l-1})\right.\\
&\qquad \left.-l-\varepsilon \sum_{j=0}^{l-1} u_j\right\} \Big/ [a(k)+b(k+l)+\varepsilon]+u_{l-1}\\
&\quad \geqslant \left\{[a(l)+l-\beta_1 l-\varepsilon]u_{l-1}+[\beta_0 \wedge a(l)](u_{l-2}-u_{l-1})\right.\\
&\qquad \left.-l-\varepsilon \sum_{j=0}^{l-1} u_j\right\} \Big/ [b(l)+\varepsilon]+u_{l-1}\\
&\quad =\left\{[\beta_0 \vee a(l)+l]u_{l-1}+[\beta_0 \wedge a(l)]u_{l-2}-l-\varepsilon \sum_{j=0}^{l-1} u_j\right\}\\
&\qquad \times [b(l)+\varepsilon]^{-1} \qquad \text{QED}
\end{aligned}$$

Now, I turn to the main discussion. For $u_l(\varepsilon) \geqslant \gamma(1+\alpha l)^{-1}$, $l=1, 2$, it suffices that

$$\bar{u}_l > \gamma(1+\alpha l)^{-1}, \qquad l=1, 2 \tag{4.6}$$

where $\bar{u}_1 = [(\delta_1 + \beta_0)/(\beta_1 + \beta_0)] \wedge 1$ and

$$\begin{aligned}\bar{u}_2 = \bar{u}_1 \qquad & \text{if} \quad (\delta_1 - \beta_1 + \delta_2 + 1)\bar{u}_1 + [\tfrac{1}{2}\beta_0 \wedge (\delta_1 + \delta_2)](1 - \bar{u}_1) \geqslant 1, \\ = (\{[\tfrac{1}{2}\beta_0 \vee (\delta_1 + \delta_2) + 1]\bar{u}_1 & + [\tfrac{1}{2}\beta_0 \wedge (\delta_1 + \delta_2)] - 1\} \\ \times (\tfrac{1}{2}\beta_0 + \beta_1)^{-1}) \vee 0 \qquad & \text{otherwise}\end{aligned}$$

For $l \geqslant 3$, by Lemma 4.6, we require that

$$\gamma \left\{ \frac{\beta_0 \vee a(l) + l - \varepsilon}{1 + \alpha(l-1)} + \frac{\beta(0) \wedge a(l) - \varepsilon}{1 + \alpha(l-2)} - \frac{b(l) + \varepsilon}{1 + \alpha l} \right\} \geqslant (1 + \varepsilon) l - 2\varepsilon \tag{4.7}$$

Let

$$l_0 = (\text{the first integer so that } a(l) \geqslant \beta_0) \vee 3$$

Then, for $l \geqslant l_0$, (4.7) becomes

$$\begin{aligned}&\gamma(\delta_1 - \beta_1 - \delta_2) + [\gamma\delta_2 - (1 + \varepsilon)\alpha] l + \gamma - 1 \\ &\quad + \gamma \left\{ \frac{\alpha[a(l)/l + 1]}{1 + \alpha(l-1)} + \frac{2\alpha\beta_0/l}{1 + \alpha(l-2)} \right\} \\ &\geqslant \frac{\gamma\alpha\varepsilon}{l} \left[\frac{1}{1 + \alpha(l-1)} + \frac{2}{1 + \alpha(l-2)} \right] + \varepsilon + \frac{\varepsilon}{l}(3\gamma - 2\alpha) - 2\varepsilon\alpha\end{aligned}$$

Hence, we require that

$$\begin{aligned}&\gamma(\delta_1 - \beta_1) + (\gamma\delta_2 - \alpha) l + \gamma - 1 \\ &\quad + \gamma \left[\frac{\alpha(1 + \delta_1) - \delta_2}{1 + \alpha(l-1)} + \frac{2\alpha\beta_0/l}{1 + \alpha(l-2)} \right] > 0, \qquad l \geqslant l_0\end{aligned} \tag{4.8}$$

Similarly, for $3 \leqslant l \leqslant l_0$, we require that

$$\begin{aligned}&\gamma(\delta_1 - \beta_1) + (\gamma\delta_2 - \alpha) l + \gamma - 1 \\ &\quad + \gamma \left[\frac{\alpha(\beta_0/l + 1)}{1 + \alpha(l-1)} + \delta_2 + \frac{2\alpha(\delta_1 + \delta_2) - 2\delta_2}{1 + \alpha(l-2)} \right] > 0\end{aligned} \tag{4.9}$$

Since

$$\frac{\alpha[a(l)/l + 1]}{1 + \alpha(l-1)} + \frac{2\alpha\beta_0/l}{1 + \alpha(l-2)} \geqslant \frac{\alpha(\beta_0/l + 1)}{1 + \alpha(l-1)} + \frac{2\alpha a(l)/l}{1 + \alpha(l-2)}$$

holds if and only if $\beta_0 \geqslant a(l)$. So does

$$\frac{\alpha(1+\delta_1)-\delta_2}{1+\alpha(l-1)}+\frac{2\alpha\beta_0/l}{1+\alpha(l-2)} \geqslant \frac{\alpha(\beta_0/l+1)}{1+\alpha(l-1)}+\delta_2+\frac{2\alpha(\delta_1+\delta_2)-2\delta_2}{1+\alpha(l-2)}$$

Hence, if we set

$$\Phi(\beta, \delta, l)=\left[\frac{\alpha(1+\delta_1)-\delta_2}{1+\alpha(l-1)}+\frac{2\alpha\beta_0/l}{1+\alpha(l-2)}\right] \wedge\left[\frac{\alpha(\beta_0/l+1)}{1+\alpha(l-1)}+\delta_2+\frac{2\alpha(\delta_1+\delta_2)-2\delta_2}{1+\alpha(l-2)}\right] \tag{4.10}$$

then (4.8) and (4.9) can be unified into

$$\gamma(\delta_1-\beta_1)+(\gamma\delta_2-\alpha)l+\gamma-1+\gamma\Phi(\beta, \delta, l)>0, \qquad l \geqslant 3 \tag{4.11}$$

Combining (4.11) with (4.6), we need that

$$[1+\delta_1-\beta_1+\delta_2 l+\Phi(\beta, \delta, l)]/(1+\alpha l) > 1/\gamma > \{[(1+\alpha)\bar{u}_1] \wedge [(1+2\alpha)\bar{u}_2]\}^{-1}, \qquad l \geqslant 3 \tag{4.12}$$

In view of the left-hand side, it is necessary that $\alpha \leqslant \delta_2$. This fact plus (4.8) suggests the choice

$$\alpha=\delta_2/(1+\delta_1)<\delta_2$$

From now on we fix this α. Then, (4.12) becomes

$$H(\beta, \delta, l) \equiv \frac{1}{1+\alpha l}[\Phi(\beta, \delta, l)-\beta_1] > (1+\delta_1)(\{[(1+\delta_1+\delta_2)\bar{u}_1]/[(1+\delta_1+2\delta_2)\bar{u}_2]\}^{-1}-1) \equiv \bar{w}, \qquad l \geqslant 3 \tag{4.13}$$

On the other hand, for $l \geqslant l_0$, $\Delta H(\beta, \delta, \cdot)(l) \geqslant 0$ if and only if

$$\Delta\Phi(\beta, \delta, \cdot)(l) \geqslant \frac{\alpha}{1+\alpha l}[\Phi(\beta, \delta, l)-\beta_1]$$

That is,

$$\begin{aligned}
\beta_1 &\geqslant \Phi(\beta, \delta, l) - \frac{1+\alpha l}{\alpha} \Delta\Phi(\beta, \delta, \cdot)(l) \\
&= \Phi(\beta, \delta, l)\left[1 + \frac{1+\alpha l}{\alpha}\left(1 - \frac{\Phi(\beta, \delta, l+1)}{\Phi(\beta, \delta, l)}\right)\right] \\
&= \Phi(\beta, \delta, l)\left[1 + \frac{1+\alpha l}{\alpha}\left(1 - \frac{l[1+\alpha(l-2)]}{(l+1)[1+\alpha(l-1)]}\right)\right] \\
&= \frac{2\alpha\beta_0}{l[1+\alpha(l-2)]}\left[1 + \frac{1+\alpha l}{\alpha(l+1)} + \frac{l(1+\alpha l)}{(l+1)[1+\alpha(l-1)]}\right] \\
&= \frac{2\alpha\beta_0}{l[1+\alpha(l-2)]}\left[1 + \frac{1}{\alpha(l+1)} + \frac{2}{l+1} + \frac{\alpha l}{(l+1)[1+\alpha(l-1)]}\right] \qquad (4.14)
\end{aligned}$$

Because the right-hand side is decreasing in l, if we let l_1 be the first integer starting from l_0 so that (4.14) holds, then

$$\inf_{l \geqslant l_0} H(\beta, \delta, l) = H(\beta, \delta, l_1)$$

Thus, (4.13) reduces to

$$\min_{3 \leqslant l \leqslant l_0 - 1} H(\beta, \delta, l) \wedge H(\beta, \delta, l_1) > \bar{w} \qquad (4.15)$$

Even though one can evaluate the above minimum by using the same argument in the case that $\delta_2 \geqslant 1$, I will not do so, since only small β_0 (i.e., small l_0) has to be considered, as we will see soon.

Next, I seek some more explicit conditions. The reason I started from $l=3$ is to leave u_0 free and so γ can be greater than one. Otherwise, $u_0(\varepsilon) \geqslant \gamma(1+\alpha\cdot 0)^{-1}$ is trivial and so we do not need to consider $\bar{u}_2$ as one of our initial points. More precisely, let $\gamma=1$. Then the condition (4.6) for $l=1$ becomes

$$\delta_1 + \delta_2 \frac{\delta_1 + \beta_0}{1+\delta_1} > \beta_1 \qquad (4.16)$$

As for (4.8), note that

$$\begin{aligned}
&\frac{\alpha}{1+\alpha(l-1)} + \delta_2 + \frac{2\alpha(\delta_2 - 1)}{1+\alpha(l-2)} \\
&\quad = \frac{\delta_2(1+\alpha l)}{1+\alpha(l-2)} + \alpha\left[\frac{1}{1+\alpha(l-1)} - \frac{2}{1+\alpha(l-2)}\right]
\end{aligned}$$

$$= \frac{1+\alpha l}{1+\alpha(l-2)} \left[\delta_2 - \frac{\alpha}{1+\alpha(l-1)} \right]$$

$$= \frac{\delta_2(1+\alpha l)}{1+\alpha(l-2)} \frac{\delta_1+\delta_2(l-1)}{1+\delta_1+\delta_2(l-1)}$$

$$= \frac{\alpha(1+\alpha l)}{[1+\alpha(l-2)][1+\alpha(l-1)]} [\delta_1+\delta_2(l-1)]$$

and

$$\frac{\alpha\beta_0}{l} \left[\frac{2}{1+\alpha(l-2)} - \frac{1}{1+\alpha(l-1)} \right] = \frac{\alpha(1+\alpha l)}{[1+\alpha(l-2)][1+\alpha(l-1)]} \frac{\beta_0}{l}$$

Hence

$$\begin{aligned} \Phi(\beta, \delta, l) &= \frac{\alpha\beta_0}{l[1+\alpha(l-1)]} + \frac{\alpha(1+\alpha l)}{[1+\alpha(l-2)][1+\alpha(l-1)]} \\ &\quad \times \left\{ \left(\frac{\beta_0}{l} \right) \wedge [\delta_1 + \delta_2(l-1)] \right\} \\ &> \frac{\alpha\beta_0}{l[1+\alpha(l-1)]} > 0 \end{aligned} \tag{4.17}$$

On the other hand,

$$\begin{aligned} &\delta_1 l + \frac{\beta_0}{l[1+\alpha(l-1)]} \\ &= \frac{\delta_1}{2} + \frac{\delta_1}{2\alpha} [1+\alpha(l-1)] + \frac{\beta_0}{l[1+\alpha(l-1)]} + \frac{\delta_1}{2\alpha} (\alpha - 1) \\ &\geqslant 3 \left[\left(\frac{\delta_1}{2} \right)^2 \frac{\beta_0}{\alpha} \right]^{1/3} + \frac{\delta_1}{2\alpha} (\alpha - 1) \end{aligned}$$

Therefore, (4.8) follows from

$$3 \left[\left(\frac{\delta_1 \delta_2}{2(1+\delta_1)} \right)^2 \beta_0 \right]^{1/3} + \frac{\delta_1(\delta_2 + 1 + \delta_1)}{2(1+\delta_1)} > \beta_1 \tag{4.18}$$

Clearly, (4.16) and (4.18) hold for large enough β_0. Certainly, instead of (4.18), we can use

$$\delta_1 + \delta_1 \alpha N_0 + \frac{\alpha \beta_0}{N_0[1 + \alpha(N_0 - 1)]} > \beta_1 \tag{4.19}$$

where N_0 is the first integer starting from 2, so that

$$\frac{\delta_1}{\beta_0} \geqslant \frac{1 + 2\alpha l}{l(l+1)[1 + \alpha(l-1)](1 + \alpha l)}$$

Now, I summarize the above discussions as follows.

Corollary 4.7. For the first Schlögl model, take $\alpha = 0$ if $\delta_1 > \beta_1$ and $\delta_2/(1+\delta_1)$ otherwise. If (4.15) holds, then we can choose some $\gamma > 0$ and $\varepsilon > 0$ such that $u_l(\varepsilon) \geqslant \gamma(1 + \alpha l)^{-1}$ for all $l \geqslant 1$ (in the case that $\delta_1 > \beta_1$, simply take $\gamma = 1$). In particular, the same assertion is true if (4.16) and one of (4.18) and (4.19) holds. Furthermore, for fixed β_1, δ_1, and δ_2, the same assertion holds for all large enough β_0.

Example. Take $\delta_1 = 1$, $\delta_2 = 2$, $\beta_1 = 9$, and $\alpha = 1$. If we take $\gamma = 1$, then for (4.16) and (4.18), we require that $\beta_0 > 4 \times (8/3)^2 \approx 76$. For (4.16) and (4.19) it is enough that $\beta_0 > 75$. However, for (4.15), we need only that $\beta_0 > 4 + (145)^{1/2} \approx 16.0416$. To show this, let $15 < \beta_0 \leqslant 18$. Then we have $l_0 = l_1 = 4$, $(4\bar{u}_1) \wedge (6\bar{u}_2) = 4\bar{u}_1 = 4(1+\beta_0)/(9+\beta_0)$, and

$$H(\beta, \delta, 3) > H(\beta, \delta, 4) = \frac{1}{5}\left(\frac{\beta_0}{6} - 9\right)$$

Hence

$$\frac{1}{5}\left[\frac{\beta_0}{6} - 9\right] > 2\left[\frac{9 + \beta_0}{4(1+\beta_0)} - 1\right]$$

gives the solution $\beta_0 > 4 + (145)^{1/2}$. Finally, if we set $\beta_0^c = \inf\{\beta_0$: the u-criterion is available for this example with pure birth rate $\beta_0\}$, then it is not difficult to show that $\beta_0^c \in (10.406, 10.4061)$. However, the u^{cm}-criterion is not applicable for any β_0.

I conclude this paper with a general result.

Theorem 4.8. Under (H), for fixed $\beta_1, \ldots, \beta_m$, $\delta_1, \ldots, \delta_{m+1}$ and large enough β_0, we have $u_l(\varepsilon) \geqslant (1 + \alpha l)^{-1}$ for some $\varepsilon, \alpha > 0$, and all $l \geqslant 0$. In other words, the reaction-diffusion processes are ergodic for all large enough β_0.

Proof. When $m=1$, the assertion with $\alpha=\delta_2/(1+\delta_1)$ was proved in Corollary 4.7. Now, assume that $m\geqslant 2$ and take $\alpha=1$. For $l\geqslant 2$, it suffices that

$$\frac{b(k)\vee a(k+l)+l}{l}+\frac{b(k)\wedge a(k+l)}{l-1}-\frac{a(k)+b(k+l)}{l+1}\geqslant(1+\varepsilon)l, \qquad k\geqslant 0$$

Equivalently,

$$\frac{\Delta_l a(k)-\Delta_l b(k)+l}{l}+\frac{b(k)\wedge a(k+l)}{l(l-1)}+\frac{a(k)+b(k+l)}{l(l+1)}\geqslant(1+\varepsilon)l, \qquad k\geqslant 0 \tag{4.20}$$

Set

$$s(k,l,j)=l^j+l^{j-1}\sum_{k-j\leqslant i_1\leqslant k} i_1+l^{j-2}\sum_{k-j\leqslant i_1<i_2\leqslant k} i_1 i_2 + \cdots + \sum_{k-j\leqslant i_1<\cdots<i_j\leqslant k} i_1\cdots i_j$$

Then we have

$$s(k,l,j-1)\leqslant s(k,l,j)/l \qquad \text{for all} \quad j\leqslant k$$

and

$$[\Delta_l a(k)-\Delta_l b(k)]/l=\sum_{j=1}^{m}(\delta_j-\beta_j)\,s(k,l,j-1)+\delta_{m+1}s(k,l,m) \tag{4.21}$$

Hence

$$\begin{aligned}
&[\Delta_l a(k)-\Delta_l b(k)]/l\\
&\quad\geqslant c\sum_{j=1}^{m}s(k,l,j-1)+\delta_{m+1}s(k,l,m)\\
&\quad\geqslant cs(k,l,m-1)(1+1/l+\cdots+1/l^{m-1})+\delta_{m+1}s(k,l,m)\\
&\quad\geqslant cls(k,l,m-1)/(l-1)+\delta_{m+1}s(k,l,m)\\
&\quad\geqslant[\delta_{m+1}+c/(l-1)]\,s(k,l,m)\\
&\quad\geqslant[\delta_{m+1}+c/(l-1)]\,l^m, \qquad k\geqslant m
\end{aligned}$$

where $c=\min\{(\delta_j-\beta_j)\wedge 0\colon 1\leqslant j\leqslant m\}$. Thus, we can choose an $L_1\geqslant 2$ independent of $k\geqslant 0$, so that

$$[\Delta_l a(k)-\Delta_l b(k)]/l\geqslant(1+\varepsilon)l \tag{4.22}$$

for all $l \geqslant L_1$ and $k \geqslant m$. On the other hand, for the finite set $\{k: 0 \leqslant k \leqslant m-1\}$, by (4.21), we can choose an L_2 so that (4.22) also holds for all $l \geqslant L_2$. Put $L_0 = L_1 \vee L_2$. Then (4.22) holds for all $l \geqslant L_0$ and $k \geqslant 0$ and so does (4.20).

Now consider the case that $2 \leqslant l \leqslant L_0 - 1$. Again, by (4.21), it follows that (4.22) holds for large enough $k \geqslant K_0$ (say!). Hence (4.20) holds for all $2 \leqslant l \leqslant L_0 - 1$ and $k \geqslant K_0$. But then we can choose β_0 large enough so that (4.20) holds for all $2 \leqslant l \leqslant L_0 - 1$ and $k \leqslant K_0$ as well. Therefore, (4.20) holds for all $l \geqslant 2$ and $k \geqslant 0$ whenever β_0 is large enough.

Finally, for the case that $l = 1$, the proof is similar and simpler. QED

ACKNOWLEDGMENTS

I thank Prof. X. F. Liu and Dr. L. P. Huang for valuable discussions. I also acknowledge the referees for their helpful comments. This project was supported by the Ying-Tung Fok Education Foundation and the NSFC.

REFERENCES

1. M. F. Chen, Infinite dimensional reaction diffusion processes, *Acta Math. Sin. N. S.* **1**(3):261–273 (1985).
2. M. F. Chen, *Jump Processes and Interacting Particle Systems* (Beijing Normal University Press, Beijing, 1986) [in Chinese].
3. M. F. Chen, Coupling for jump processes, *Acta Math. Sin. N. S.* **2**:123–136 (1986).
4. M. F. Chen, Existence theorems for interacting particle systems with non-compact state space, *Sci. Sin. A* **30**:148–156 (1987).
5. M. F. Chen, Stationary distributions for infinite particle systems with non-compact state space, *Acta Math. Sci.* **9**:7–19 (1989).
6. M. F. Chen, Couplings of jump processes, *Chin. Ann. of Math.*, to appear.
7. W. D. Ding, R. Durrett, and T. M. Liggett, Ergodicity of reversible reaction diffusion processes, preprint, Cornell University, Ithaca, New York (1988).
8. W. D. Ding and X. G. Zheng, Ergodic theorems for linear growth processes with diffusion, *Chin. Ann. of Math.* **10B**(3):386–402 (1989).
9. D. Han, Existence and uniqueness of solution to the Martingale problem for the multi-species infinite dimensional reaction diffusion particle systems, Ph. D. thesis, Beijing Normal University, Beijing (1989).
10. C. Neuhauser, An ergodic theorem for Schlögl models with small migration, preprint, Cornell University, Ithaca, New York (1989).