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# INFINITE DIMENSIONAL REACTION-DIFFUSION PROCESSES

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## §1. Introduction

The background of the reaction-diffusion processes can be found in Haken[4] and Yan and Lee[11]. For finite dimensional case, the reaction-diffusion processes have been studied by Yan and Chen[10]. In particular, it has been proved there that the process corresponding to Schlögl model is always ergodic, hence the invariant measure is unique so there is no phase transition. However, many physists think there exists the phase transition for Schlögl model. Thus, the phase transition may be appeared in the infinite dimensional case. This is one of the reasons why we are interested in this area.

Let $S$ be a countable set. Imagining each $u \in S$ as a small vessel in which there is a reaction and suppose there are some diffusions between the vessels. This is so called a reaction-diffusion process. If there is no reaction in each vessel, it is just the zero-range process which was first introduced and studied by Spitzer (1970)[9]. For a special case, Holley (1970)[5] proved the existence for the zero-range process. and a general existence theorem was obtained by Liggett (1973)[7]. Recently, Andjel (1982)[1] has given a simpler proof for Liggett's result by using the method developed by Liggett and Spitzer in [8]. More recently, Zheng and Ding[12] have obtained an existence result in the case that the reactions are all birth-death processes with linear rate function. The purpose of this paper is to prove a general existence theorem (Theorem (1.1)) for the general infinite dimensional reaction- diffusion process.

Throughout the paper we consider only a single reactant. In each $u \in S$, the number of particles of the reactant is evaluated in $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$.

The rate function of the reaction in $u$ can be described by a $Q$-matrix $Q_u = (q_u(i,j) : i, j \in \mathbb{Z}_+)$. We use a transition probability function $P = (p(u,v) : u, v \in S)$ to describe the diffusions between the vessels. Thus if there are $k$ particles in $u$, then the rate function of the diffusion from $u$ to $v$ is $C_u(k) \cdot p(u,v)$, where

$$C_u \geqslant 0, \quad C_u(0) = 0, \qquad u \in S. \tag{1}$$

As in [7], we will use

$$\mathscr{E} = \left\{\eta \in \mathbb{Z}_+^S : \|\eta\| := \textstyle\sum_{x\in S} \eta(x)\alpha(x) < \infty\right\}$$

as our state space instead of $E := Z_+^S$, where $\alpha$ is a positive function on $S$ such that

$$\sum_{y\in S} p(x,y)\alpha(y) \leqslant M\alpha(x), \qquad x \in S, \tag{2}$$

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and $M$ is a positive constant. For a given $M \geqslant 1$ and $P$, such $\alpha$ exists always[8]. From now on, we will fix such $\alpha$. $\mathscr{E}$ will be endowed with the smallest $\sigma$-algebra for which the map $\eta \to \eta(x)$ is measurable for each $x \in S$.

Let $\mathscr{L}$ be the class of Lipschitz functions $f$ on $\mathscr{E}$. Those are the ones for which there is a constant $C$ such that

$$|f(\eta) - f(\zeta)| \leqslant C\|\eta - \zeta\|, \qquad \eta, \zeta \in \mathscr{E},$$

where $\|\eta - \zeta\| = \sum_x |\eta(x) - \zeta(x)|\alpha(x)$. Then $L(f)$ is defined to be the smallest such constant $C$.

Now, we are at the position to write formally the generator for our reaction-diffusion process:

$$\begin{aligned}\Omega f(\eta) = &\sum_{u \in S}\sum_{k \neq 0} q_u(\eta(u), \eta(u)+k)[f(\eta + ke_u) - f(\eta)] \\ &+ \sum_{u \in S} C_u(\eta(u)) \sum_{v \in S}[f(\eta - e_u + e_v) - f(\eta)], \qquad \eta \in \mathscr{D}\end{aligned} \tag{3}$$

where $e_u$ is the element in $E$ whose value corresponding to $u$ is one, and other values are zero, and

$$\mathscr{D} = \Big\{\eta \in \mathscr{E} : |||\eta||| := \sum_{u \in S, \eta(u) \neq 0} \sum_{k \neq 0} q_u(\eta(u), \eta(u)+k)|k|\alpha(u) < \infty\Big\},$$

where, and elsewhere, we use the following convention:

$$q_u(i, j) = 0, \qquad i \in \mathbb{Z}_+,\ j \notin \mathbb{Z}_+,\ u \in S.$$

In the case of $S$ being finite, $\Omega$ corresponds to a totally stable and conservative $Q$-matrix (such are all the $Q$-matrices considered in this paper). Also, we suppose that

$$K = \sup_{u,k} |C_u(k) - C_u(k+1)| < \infty; \tag{4}$$

$$\|\beta\| := \sum_u \beta(u)\alpha(u) = \sum_u \alpha(u) \sum_{k=1}^{\infty} q_u(0,k)k < \infty; \tag{5}$$

$$\sum_{k \neq 0} q_u(i, i+k)|k| < \infty, \qquad i \in \mathbb{Z}_+,\ u \in S. \tag{6}$$

When $S$ is finite, we will use a oupling of two reaction-diffusion processes, and its generator is:

$$\overline{\Omega} f(\eta_1, \eta_2) = \sum_{u \in S} \overline{Q}_u f(\eta_1, \eta_2) + \overline{D} f(\eta_1, \eta_2), \qquad \eta_1,\ \eta_2 \in S, \tag{7}$$

where

$$\begin{aligned}\overline{Q}_u f(\eta_1, \eta_2) = &\sum_{k \neq 0} q_u(\eta_1(u), \eta_1(u)+k)[f(\eta_1 + ke_u, \eta_2) - f(\eta_1, \eta_2)] \\ &\times I_{\Delta_u^c}(\eta_1(u), \eta_2(u)) \\ &+ \sum_{k \neq 0} q_u(\eta_2(u), \eta_2(u)+k)[f(\eta_1, \eta_2 + ke_u) - f(\eta_1, \eta_2)] \\ &\times I_{\Delta_u^c}(\eta_1(u), \eta_2(u)) \\ &+ \sum_{k \neq 0} \big(q_u(\eta_1(u), \eta_1(u)+k) - q_u(\eta_2(u), \eta_2(u)+k)\big)^+ \\ &\times [f(\eta_1 + ke_u, \eta_2) - f(\eta_1, \eta_2)] I_{\Delta_u}(\eta_1(u), \eta_2(u))\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\neq 0}\big(q_u(\eta_2(u),\eta_2(u)+k)-q_u(\eta_1(u),\eta_1(u)+k)\big)^+\\
&\quad\times[f(\eta_1,\eta_2+ke_u)-f(\eta_1,\eta_2)]I_{\Delta_u}(\eta_1(u),\eta_2(u))\\
&+\sum_{k\neq 0}\big(q_u(\eta_1(u),\eta_1(u)+k)\wedge q_u(\eta_2(u),\eta_2(u)+k)\big)\\
&\quad\times[f(\eta_1+ke_u,\eta_2+ke_u)-f(\eta_1,\eta_2)]I_{\Delta_u}(\eta_1(u),\eta_2(u))
\end{aligned}
$$

$$
\begin{aligned}
\overline{D}f(\eta_1,\eta_2)&=\sum_{u\in S}[C_u(\eta_1(u))-C_u(\eta_2(u))]^+\sum_{v\in S}p(u,v)\\
&\quad\times[f(\eta_1-e_u+e_v,\,\eta_2)-f(\eta_1,\eta_2)]\\
&=\sum_{u\in S}[C_u(\eta_2(u))-C_u(\eta_1(u))]^+\sum_{v\in S}p(u,v)\\
&\quad\times[f(\eta_1,\,\eta_2-e_u+e_v)-f(\eta_1,\eta_2)]\\
&=\sum_{u\in S}[C_u(\eta_1(u))\wedge C_u(\eta_2(u))]\sum_{v\in S}p(u,v)\\
&\quad\times[f(\eta_1-e_u+e_v,\,\eta_2-e_u+e_v)-f(\eta_1,\eta_2)],
\end{aligned}
$$

where $\Delta_u(u\in S)$ is the except set with respect to the order relation:

$$
\begin{aligned}
\Delta_u=\{&(i_1,i_2)\in\mathbb{Z}_+\times\mathbb{Z}_+:\ \text{if } i_1\geqslant i_2,\ \text{then either there exists a}\\
&j_1<i_2 \text{ such that } q_u(i_1,j_1)>0,\ \text{or there exists a } j_2>i_1 \text{ such}\\
&\text{that } q_u(i_2,j_2)>0;\ \text{if } i_1\leqslant i_2,\ \text{then either there exists a } j_1>i_2\\
&\text{such that } q_u(i_1,j_1)>0,\ \text{or there exists a } j_2<i_1 \text{ such that}\\
&q_u(i_2,j_2)>0\}.
\end{aligned}
$$

It is clear that $\Delta_u(u\in S)$ is symmetric in the sense that either both $(i_1,i_2)$ and $(i_2,i_1)$ belong to $\Delta_u$, or neither of them belongs to $\Delta_u$.

Set

$$
q_u^{(N)}(i,i+k)=\begin{cases} q_u(i,i+k), & 0\leqslant i\leqslant N\\ q_u(N,N+k), & i\geqslant N, \qquad k\neq 0,\ u\in S\end{cases} \tag{8}
$$

Denote by $\Delta_u^{(N)}$ the except set with respect to the order relation for $Q_u^{(N)}=\big\{q_u^{(N)}(i,j):i,j\in\mathbb{Z}_+\big\}$. We suppose that there exists an $N_0\geqslant 1$ such that

$$
\Delta_u^{(N)}\subset\Delta_u,\qquad N\geqslant N_0,\ u\in E. \tag{9}
$$

A $Q$-matrix (resp. $\Omega$) is called regular if it determines a unique $Q$-process. Since a general uniqueness criterion for $Q$-processes was obtained (c.f. [6]), and more practical sufficient conditions for the uniqueness for multi-dimensional $Q$-processes have been obtained in [10], we can pay our attention to the infinite dimensional oases. Therefore, we will assume that the $\Omega$ and $\overline{\Omega}$ defined in (3) and (7), respectively, are regular in the case of $S$ being finite.

Now, we put

$$
\begin{aligned}
g_u(j_1,j_2)&=\sum_{k\neq 0}\big(q_u(j_2,j_2+k)-q_u(j_1,j_1+k)\big)k(j_2-j_1)^{-1},\\
h_u(j_1,j_2)&=2\sum_{k=1}^{\infty}\big[\big(q_u(j_2,j_1-k)-q_u(j_1,2j_1-j_2-k)\big)^+\\
&\quad+\big(q_u(j_1,j_2+k)-q_u(j_2,2j_2-j_1+k)\big)^+\big]k(j_2-j_1)^{-1},\\
&\qquad j_2\geqslant j_1\geqslant 0.
\end{aligned} \tag{10}
$$

The following condition is important[1]:

$$K_2' = \sup\{g_u(j_1,j_2) + h_u(j_1,j_2) I_{\Delta_u}(j_1,j_2):\ \ u\in S,\ j_2>j_1\geqslant 0\} < \infty. \tag{11}$$

Next we put

$$K_2'' = \sup\left\{\frac{C_u(j_1)-C_u(j_2)}{j_2-j_1}:\ u\in S,\ j_2>j_1\geqslant 0\right\}$$

and

$$K_1 = \sup\{g_u(0,j_2): u\in S, j_2\geqslant 1\}. \tag{12}$$

It is now obvious that

$$K_1\leqslant K_2'<\infty;\qquad K_2''\leqslant K<\infty.$$

Finally, we set

$$K_2 = K_2' + K_2''. \tag{13}$$

Now choose $\Lambda_n$ to be finite sets of $S$, increasing to $S$, and define

$$p_n(u,v) = \begin{cases} p(u,v), & u,v\in\Lambda_n,\ u\neq v,\\ 1, & u=v\notin\Lambda_n\\ p(u,v)+\sum_{w\notin\Lambda_n} p(u,w), & u=v\in\Lambda_n.\end{cases}$$

In (3) and (7), replacing $P$ and $S$ with $P_n := (p_n(u,v))$ and $\Lambda_n$, one gets the operators $\Omega_n$ and $\overline{\Omega}_n$, respectively.

The following theorem is the main result of this paper:

**Theorem 1.1.** Suppose that the conditions (4), (5), (6), (9) and (11) hold, and $\overline{\Omega}_n$ is regular for each $n\geqslant 1$. Then, there exists a semigroup $S(t)$ of operators on $\mathscr{L}$, such that $S(0)=I$, and $S(t)$ is strongly contraction on the uniform closure $\overline{\mathscr{L}}$ of $\mathscr{L}$. Moreover, for every $f\in\mathscr{L}$, the semigroup satisfies the following properties:

$$|S(t)f(\eta)-S(t)f(\zeta)|\leqslant L(f)\|\eta-\zeta\|\, e^{t(K_2+K(M+1))},\qquad \eta,\ \zeta\in\mathscr{E}; \tag{14}$$

$$S(t)f(\eta) = f(\eta)+\int_0^t \Omega S(s)f(\eta)\mathrm{d}s,\qquad \eta\in\mathscr{D}; \tag{15}$$

$$\begin{aligned}|S(t)f(\eta)-f(\eta)|\leqslant{}& L(f)\big(K_2+K(M+1)\big)^{-1}\big(|||\eta|||+\|\beta\|+K(M+1)\|\eta\|\big)\\ &\times\big(\exp\big[t\big(K_2+K(M+1)\big)\big]-1\big),\qquad \eta\in\mathscr{D};\end{aligned} \tag{16}$$

$S(t)f(\eta)$ is continuous in $t$ for each $\eta\in\mathscr{E}$; (17)

$\Omega S(t)f(\eta)$ is continuous in $t$ for each $\eta\in\mathscr{E}$; (18)

$$\lim_{t\downarrow 0}\frac{S(t)f(\eta)-f(\eta)}{t} = \Omega f(\eta),\qquad \eta\in\mathscr{D}. \tag{19}$$

Also, there exists a Markov process $(\{\eta_t\}_{t\geqslant 0},\mathbb{P}^\eta)$ evaluated in $\mathscr{E}$ such that

$$S(t)f(\eta)=\mathbb{E}^\eta f(\eta_t)=\int f(\xi)\mathbb{P}^\eta[\eta_t\in\mathrm{d}\xi],\qquad f\in\mathscr{L},\ \eta\in\mathscr{E}. \tag{20}$$

[1] If we take $S=\{u\}$, i.e. $|S|=1$, and take

$$q_u(i,j)=\begin{cases}\lambda_1 i^2, & j=i+1,\ i\geqslant 0,\\ \lambda_2 i^2, & j=i-1,\ i\geqslant 1,\\ 0, & \text{other } j\neq i\end{cases}$$

then it is easy to check that $K_2'<\infty$ iff the $Q$-matrix $Q_u=(q_u(i,j))$ is regular.

This theorem will be proved in §3. In §2, we will establish some basis estimates which are the keys to this paper. To conclude this section, we would like to show two examples:

Schlögl Model[4,10]. Take

$$q_u(i,j)=\begin{cases}\lambda_1 a_u\binom{i}{2}+\lambda_4 b_u, & j=i+1,\ i\geqslant 0,\\ \lambda_2\binom{i}{3} & j=i-1,\ i\geqslant 1,\\ 0, & \text{other } i\neq j,\end{cases}$$

and where $\lambda_1,\cdots,\lambda_4,a_u,b_u>0$. Then, the condition (5) becomes $\sum_u b_u\alpha(u)<\infty$. It was proved in [10; Theorem 4, Theorem 5 and Proposition 1] that $\Omega_n$ and $\overline{\Omega}_n$, are regular for each $n\geqslant 1$. Since $\Delta_n=\{(i_1,i_2):i_1=i_2\}$, we need only to cheek the condition (11). But in this case,

$$\begin{aligned}g_u(j_1,j_2)=&\Big[\lambda_1 a_u\binom{j_2}{2}+\lambda_4 b_u-\lambda_2\binom{j_2}{3}-\lambda_3 j_2\\ &-\lambda_1 a_u\binom{j_1}{2}-\lambda_4 b_u+\lambda_3\binom{j_1}{3}+\lambda_3 j_1\Big](j_2-j_1)^{-1}\\ =&\frac{1}{2}(\lambda_1 a_u+\lambda_2)(j_1+j_2)-\left(\frac{\lambda_1 a_u}{2}+\frac{\lambda_3}{3}+\lambda_3\right)-\frac{\lambda_2}{6}(j_1^2+j_1 j_2+j_2^2),\end{aligned}$$

hence

$$K_2'=\sup\{g_u(j_1,j_2):j_2>j_1\geqslant 0,u\in S\}<\infty,$$

if $\sup(a_u\vee b_u)<+\infty$. Thus, if

$$\sum_u b_u\alpha(u)<+\infty\qquad\text{and}\qquad\sup_u(a_u\vee b_u)<+\infty,$$

then the Schlögl model satisfies the assumptions of Theorem 1.1, and therefore the reaction-diffusion process corresponding to the Schlögl model exists.

The second example is "An autocatalytic production of a chemical $X$"[4,10] for which

$$q_u(i,j)=\begin{cases}\lambda_1 a_u i, & j=i+1,\ i\geqslant 0,\\ \lambda_2\binom{i}{2}, & j=i-1,\ i\geqslant 1,\\ 0, & \text{other } i\neq j,\end{cases}$$

$$\lambda_1,\lambda_2,a_u>0,\quad u\in S,$$

$$C_u(k)=k,\quad k\geqslant 0,\ i\in S.$$

Then

$$K_2'=\lambda_1\sup_u a_u,$$

hence, if $\sup_u a_u<\infty$, the reaction-diffusion process also exists.

## §2. Basic Estimates

The condition (9) will be used since Remark 2.4. For the first two propositions in this section, the regularity assumptions for $\Omega_n$ and $\overline{\Omega}_n$ $(n\geqslant 1)$ are not used. Except these, we will assume the assumptions of Theorem 1.1 hold everywhere.

**Proposition 2.1.** Let $S$ be finite. Denote the minimal $Q$-process corresponding to $\Omega$ by $P(t,\eta,\zeta)$. Then we have

$$\sum_{\zeta} P(t,\eta,\zeta)\zeta(y) \leqslant \sum_{x} \big(\eta(x)e^{tK_1} + C\beta(x)e^{tK_3}\big) \sum_{n=0}^{\infty} \frac{(Kt)^n}{n!} P^{(n)}(x,y). \tag{21}$$

for each $y \in S$, $\eta \in E$ and $t \geqslant 0$ [2], where $C$ and $K_3$ are constants satisfying:

$$\begin{aligned} &C = K_3 = 0, &&\text{if} \quad \beta = 0\\ &C > 0,\ K_3 \geqslant K_1,\ CK_3 \geqslant 1, &&\text{if} \quad \beta \neq 0. \end{aligned} \tag{22}$$

*Proof.* Denote the left side and the right side of (21) by $x(t,\eta,y)$ and $\tilde{x}(t,\eta,y)$, respectively. Also, denote the $Q$-matrix corresponding to $\Omega$ by $(q(\eta,\zeta))$. Since $\{P(t,\eta,\zeta) : \eta \in E\}$ is the minimal nonnegative solution to the following equation:

$$P(t,\eta,\zeta) = \sum_{\xi\neq\eta} q(\eta,\xi) \int_0^t e^{-q(\eta)(t-s)} P(s,\xi,\zeta)\mathrm{d}s + \delta(\eta,\zeta)e^{-q(\eta)t},$$

it follows from [3; Theorem 9] that $\{x(t,\eta,y) : \eta \in E\}$ is the minimal nonnegative solution to the following equation:

$$x(t,\eta,y) = \sum_{\xi\neq\eta} q(\eta,\xi) \int_0^t e^{-q(\eta)(t-s)} x(s,\xi,y)\mathrm{d}s + \eta(y)e^{-q(\eta)t},$$

hence, by [3; Theorem 6], we need only to prove that $\{\tilde{x}(t,\eta,y) : \eta \in E\}$ satisfies:

$$\tilde{x}(t,\eta,y) \geqslant \sum_{\xi\neq\eta} q(\eta,\xi) \int_0^t e^{-q(\eta)(t-s)} \tilde{x}(s,\xi,y)\mathrm{d}s + \eta(y)e^{-q(\eta)t}.$$

If we consider $\eta$, $\beta := (\beta(x) : x \in S)$ and $\tilde{x}(t,\eta,\cdot) = \{\tilde{x}(t,\eta,y) : y \in S\}$ as column vectors, then

$$\tilde{x}(t,\eta,\cdot) = \exp[t(K_1 I + KP^*)]\,\eta + C\exp[t(K_3 I + KP^*)]\,\beta,$$

where $P^*$ is the transpose of $P$. Set

$$\begin{aligned} B_1 &= K_1 I + q(\eta)I + KP^*,\\ B_2 &= K_3 I + q(\eta)I + KP^*. \end{aligned}$$

Now, the proof reduces to showing that

$$e^{tB_1}\eta + Ce^{tB_2}\beta \geqslant \sum_{\xi\neq\eta} q(\eta,\xi) \int_0^t \big(e^{sB_1}\xi + Ce^{sB_2}\beta\big)\mathrm{d}s + \eta.$$

---

[2] If we take

$$q_u(i,j) = \begin{cases} \lambda_1 i, & j = i+1,\ i \geqslant 0\\ \lambda_2 i, & j = i-1,\ i \geqslant 1,\ \lambda_1,\lambda_2 > 0\\ 0, & \text{other } j \neq i, \end{cases}$$
$$C_u(i) = i, \qquad i,j \in \mathbb{Z}_+,\ u \in S,$$

then, in (21), "$\leqslant$" can be replaced by "$=$".

But this is obvious when $t = 0$, hence we need only to check that

$$e^{tB_1}B_1\eta + Ce^{tB_2}B_2\beta \geqslant \sum_{\xi\neq\eta} q(\eta,\xi)\big(e^{tB_1}\xi + Ce^{tB_2}\beta\big),$$

or equivalently,

$$\begin{aligned}&e^{tB_1}(K_1 I + KP^*)\eta + Ce^{tB_2}(K_3 I + KP^*)\beta\\ &\geqslant e^{tB_1}\sum_u\sum_{k\neq 0} q_u(\eta(u),\eta(u)+k)ke_u - \beta + e^{tB_1}\sum_u C_u(\eta(u))\Big[\sum_v p(u,v)e_v - e_u\Big] + e^{tB_1}\beta.\end{aligned}$$

Thus, the estimate (21) follows from checking the following three inequalities:

$$\sum_u\sum_{k\neq 0} q_u(\eta(u),\eta(u)+k)ke_u - \beta \leqslant K_1\eta, \tag{23}$$

$$\sum_u C_u(\eta(u))\Big[\sum_v p(u,v)e_v - e_u\Big] \leqslant KP^*\eta, \tag{24}$$

$$e^{tB_1}\beta \leqslant Ce^{tB_2}(K_3 I + KP^*)\beta. \quad \square \tag{25}$$

**Proposition 2.2.** Let $S$ be finite. Denote the minimal $Q$-process corresponding to $\overline{\Omega}$ by $\overline{P}(t,(\eta_1,\eta_2),(\zeta_1,\zeta_2))$. Then, for each $t \geqslant 0$ and $(\eta_1,\eta_2) \in E\times E$, we have[3]

$$\begin{aligned}&\sum_{\zeta_1,\zeta_2\in E} \overline{P}(t,(\eta_1,\eta_2),(\zeta_1,\zeta_2))\,\|\zeta_1-\zeta_2\|\\ &\leqslant \|\eta_1-\eta_2\|\exp[t(K_2+KM)] =: \bar{x}(t,\eta_1,\eta_2).\end{aligned} \tag{26}$$

*Proof.* Put $M_1 = K_2 + KM$ and denote the $Q$-matrix corresponding to $\overline{\Omega}$ by $(\bar{q}((\eta_1,\eta_2),(\zeta_1,\zeta_2)))$. As in the proof of Proposition 2.1, it is enough to show that

$$\begin{aligned}\bar{x}(t,\eta_1,\eta_2) \geqslant &\sum_{(\xi_1,\xi_2)\neq(\eta_1,\eta_2)} \bar{q}((\eta_1,\eta_2),(\xi_1,\xi_2))\int_0^t e^{-q(\eta_1,\eta_2)(t-s)}\bar{x}(s,\xi_1,\xi_2)\mathrm{d}s\\ &+\|\eta_1-\eta_2\|\,e^{-q(\eta_1,\eta_2)t}.\end{aligned}$$

This is obvious when $t = 0$, so we need only to check that

$$M_1\|\eta_1-\eta_2\| \geqslant \sum_{(\xi_1,\xi_2)\neq(\eta_1,\eta_2)} \bar{q}((\eta_1,\eta_2),(\xi_1,\xi_2))\big[\|\xi_1-\xi_2\| - \|\eta_1-\eta_2\|\big]. \tag{27}$$

First, we estimate the diffusion part:

$$\begin{aligned}J_1 = &\sum_u \big[C_u(\eta_1(u)) - C_u(\eta_2(u))\big]^+ \sum_{v\neq u} P(u,v)\\ &\times[\|\eta_1 - e_u + e_v - \eta_2\| - \|\eta_1-\eta_2\|]\\ &+\sum_u \big[C_u(\eta_2(u)) - C_u(\eta_1(u))\big]^+ \sum_{v\neq u} P(u,v)\\ &\times[\|\eta_1 - \eta_2 + e_u - e_v\| - \|\eta_1-\eta_2\|]\end{aligned}$$

[3]The footnote for (21) is also available for (28).

$$
\begin{aligned}
=&\sum_u \alpha(u)\Big\{\big[C_u(\eta_1(u))-C_u(\eta_2(u))\big]^+\mathrm{sgn}(\eta_2(u)-\eta_1(u))\\
&\quad+\big[C_u(\eta_2(u))-C_u(\eta_1(u))\big]^+\mathrm{sgn}(\eta_1(u)-\eta_2(u))\Big\}\\
&+\sum_u\big[C_u(\eta_1(u))-C_u(\eta_2(u))\big]^+\sum_v P(u,v)\alpha(v)\mathrm{sgn}(\eta_1(v)-\eta_2(v))\\
&+\sum_u\big[C_u(\eta_2(u))-C_u(\eta_1(u))\big]^+\sum_v P(u,v)\alpha(v)\mathrm{sgn}(\eta_2(v)-\eta_1(v))\\
\leqslant&(K_2''+KM)\|\eta_1-\eta_2\|.
\end{aligned}
$$

Next we estimate the reaction part. Set

$$
\begin{aligned}
J_2(u)=&\sum_{k\neq 0} q_u(\eta_1(u),\eta_1(u)+k)[\|\eta_1+ke_u-\eta_2\|-\|\eta_1-\eta_2\|]\\
&+\sum_{k\neq 0} q_u(\eta_2(u),\eta_2(u)+k)[\|\eta_2+ke_u-\eta_1\|-\|\eta_1-\eta_2\|]
\end{aligned}
$$

and assume that $\eta_1(u)>\eta_2(u)$. By an elementary calculation, we get

$$
\begin{aligned}
J_2(u)=&\sum_{k\neq 0}\big[q_u(\eta_1(u),\eta_1(u)+k)-q_u(\eta_2(u),\eta_2(u)+k)\big]k\alpha(u)\\
&+2\sum_{k\neq 0}\big[q_u(\eta_1(u),\eta_2(u)-k)+q_u(\eta_2(u),\eta_1(u)+k)\big]k\alpha(u).
\end{aligned}
$$

On the other hand, the above second term should vanish whenever $\eta_1(u)>\eta_2(u)$ and $(\eta_1(u),\eta_2(u))\notin\Delta_u$. Hence, we get

$$
\begin{aligned}
J_2(u)\leqslant&\sum_{k\neq 0}\big[q_u(\eta_1(u),\eta_1(u)+k)-q_u(\eta_2(u),\eta_2(u)+k)\big]k\alpha(u),\\
&\qquad\qquad \eta_1(u)>\eta_2(u),\ (\eta_1(u),\eta_2(u))\notin\Delta_u.
\end{aligned}
$$

Similarly, if we put

$$
\begin{aligned}
J_3(u)=&\sum_{k\neq 0}\big[q_u(\eta_1(u),\eta_1(u)+k)-q_u(\eta_2(u),\eta_2(u)+k)\big]^+[\|\eta_1+ke_u-\eta_2\|-\|\eta_1-\eta_2\|]\\
&+\sum_{k\neq 0}\big[q_u(\eta_2(u),\eta_2(u)+k)-q_u(\eta_1(u),\eta_1(u)+k)\big]^+[\|\eta_2+ke_u-\eta_1\|-\|\eta_1-\eta_2\|]
\end{aligned}
$$

and assume that $\eta_1(u)>\eta_2(u)$, then we have

$$
\begin{aligned}
J_3(u)=&\sum_{k\neq 0}\big[q_u(\eta_1(u),\eta_1(u)+k)-q_u(\eta_2(u),\eta_2(u)+k)\big]k\alpha(u)\\
&+2\sum_{k=1}^{\infty}\Big\{\big[q_u(\eta_1(u),\eta_2(u)-k)-q_u(\eta_2(u),2\eta_2(u)-\eta_1(u)-k)\big]^+\\
&\quad+\big[q_u(\eta_2(u),\eta_1(u)+k)-q_u(\eta_1(u),2\eta_1(u)-\eta_2(u)+k)\big]^+\Big\}k\alpha(u).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&J_2(u)I_{\Delta_u^c}(\eta_1(u),\eta_2(u))+J_3(u)I_{\Delta_u}(\eta_1(u),\eta_2(u))\\
&\quad\leqslant K_2'(\eta_1(u)-\eta_2(u))\alpha(u),
\end{aligned}
$$

whenever $\eta_1(u) > \eta_2(u)$. Since $J_2(u)$, $J_3(u)$ and $\Delta_u$ are all symmetric with respect to $(\eta_1(u), \eta_2(u))$, it follows that

$$\begin{aligned}&J_2(u)I_{\Delta_u^c}(\eta_1(u),\eta_2(u)) + J_3(u)I_{\Delta_u}(\eta_1(u),\eta_2(u))\\ &\quad\leqslant K_2'(\eta_2(u)-\eta_1(u))\alpha(u),\end{aligned}$$

whenever $\eta_2(u) > \eta_1(u)$. Combining the above estimates, we get

$$\begin{aligned}&\sum_{(\xi_1,\xi_2)\neq(\eta_1,\eta_2)} \bar{q}((\eta_1,\eta_2),(\xi_1,\xi_2))\big[\|\xi_1-\xi_2\| - \|\eta_1-\eta_2\|\big]\\ &\leqslant J_1 + \sum_{u\in S} J_2(u)I_{\Delta_u^c}(\eta_1(u),\eta_2(u)) + \sum_{u\in S} J_3(u)I_{\Delta_u}(\eta_1(u),\eta_2(u))\\ &\leqslant KM\|\eta_1-\eta_2\| + K_2\|\eta_1-\eta_2\| = M_1\|\eta_1-\eta_2\|.\end{aligned}$$

This proves (27). □

**Corollary 2.3.** Let $S$ be finite. Under the assumptions of Theorem 1.1, $S(t)f \in \mathscr{L}$ and

$$L(S(t)f) \leqslant L(f)\exp[t(K_2+KM)]. \tag{28}$$

*Proof.* Let $f(\eta_1,\eta_2) = f(\eta_1) - f(\eta_2)$ and $h(\eta_1,\eta_2) = \|\eta_1-\eta_2\|$. Denote the semigroup corresponding to $\overline{\Omega}$ by $\overline{S}(t)$. It follows from [2] and Proposition 2.2 that

$$\begin{aligned}&|S(t)f(\eta_1) - S(t)f(\eta_2)| = |\overline{S}(t)f(\eta_1,\eta_2)|\\ &\quad\leqslant L(f)|\overline{S}(t)h(\eta_1,\eta_2)| \leqslant L(f)\|\eta_1-\eta_2\|\exp[t(K_2+KM)].\end{aligned}$$

Hence

$$|S(t)f(\eta) - S(t)f(\zeta)| \leqslant \|\eta-\zeta\| L(f)\exp[t(K_2+KM)], \qquad f\in\mathscr{L},\ \eta,\zeta\in E. \quad \square \tag{29}$$

**Remark 2.4.** If we replace $S$ and $\Omega$ by $\Lambda_n$ and $\overline{\Omega}_n$, respectively, then we should use $M+1$ instead of $M$. Since the corresponding $K_1^{(n)}$, $K_2^{(n)'}$ and $K_2^{(n)''}$ satisfy

$$K_1^{(n)} \leqslant K_1, \quad K_2^{(n)'} \leqslant K_2', \quad K_2^{(n)''} \leqslant K_2'', \qquad n\geqslant 1,$$

the estimates (21) and (29) are available for $\Omega_n$ ($n \geqslant 1$) without changing the constants $K, K_1, K_2, K_3$ and $C$.

Now, if we replace $Q_u$ by $Q_u^{(N)} = (q_u^{(N)}(i,j))$ defined by (8), and $C_u$ by $C_u^{(N)}$:

$$C_u^{(N)}(k) = C_u(k)\wedge N, \qquad k\geqslant 0,\ N\geqslant 1,\ u\in S, \tag{30}$$

we can define $K(N)$, $K_1(N)$, $K_2'(N)$, and $K_2''(N)$ according to (4), (11) and (12), respectively. But $K(N) \leqslant K$. If $0 \leqslant j_1 < N \leqslant j_2$ and $N \geqslant N_0$, then

$$\begin{aligned}
&g_u^{(N)}(j_1, j_2) + h_u^{(N)}(j_1, j_2) \\
&\quad = \sum_{k\neq 0} \big[q_u^{(N)}(j_2, j_2+k) - q_u^{(N)}(j_1, j_1+k)\big] k (j_2 - j_1)^{-1} \\
&\qquad + 2 I_{\Delta_u^{(N)}}(j_1, j_2) \sum_{k=j_2-j_1+1}^{\infty} \big[\big(q_u^{(N)}(j_2, j_2-k) - q_u^{(N)}(j_1, j_1-k)\big)^+ \\
&\qquad + \big(q_u^{(N)}(j_2, j_2+k) - q_u^{(N)}(j_1, j_1+k)\big)^+\big](j_1 - j_2 + k)(j_2 - j_1)^{-1} \\
&\quad = \sum_{k\neq 0} \big[q_u(N, N+k) - q_u(j_1, j_1+k)\big] k (j_2 - j_1)^{-1} \\
&\qquad + 2 I_{\Delta_u^{(N)}}(j_1, j_2) \sum_{k=j_2-j_1+1}^{\infty} \big[\big(q_u(N, N-k) - q_u(j_1, j_1-k)\big)^+ \\
&\qquad + \big(q_u(N, N+k) - q_u(j_1, j_1+k)\big)^+\big](j_1 - j_2 + k)(j_2 - j_1)^{-1} \\
&\quad \leqslant \bigg\{ \sum_{k\neq 0} \big[q_u(N, N+k) - q_u(j_1, j_1+k)\big] k (j_2 - j_1)^{-1} \\
&\qquad + 2 I_{\Delta_u^{(N)}}(j_1, j_2) \sum_{k=N-j_1+1}^{\infty} \big[\big(q_u(N, N-k) - q_u(j_1, j_1-k)\big)^+ \\
&\qquad + \big(q_u(N, N+k) - q_u(j_1, j_1+k)\big)^+\big](j_1 - N + k) \bigg\}^+ (N - j_1)^{-1} \\
&\quad = \big[g_u(j_1, N) + I_{\Delta_u}(j_1, N) h_u(j_1, N)\big]^+ = K_2' \vee 0.
\end{aligned}$$

In the last step but one, we have used the following property:

$$(j_1, j_2) \in \Delta_u^{(N)} \text{ and } 0 \leqslant j_1 < N \leqslant j_2 \Longrightarrow (j_1, N) \in \Delta_u,\ N \geqslant N_0.$$

This can be checked by using (9). On the other hand, the above estimate is trivial in the case either $0 \leqslant j_1 < j_2 \leqslant N$ or $N \leqslant j_1 < j_2$, and therefore

$$K_2'(N) \leqslant K_2' \vee 0, \qquad N \geqslant N_0.$$

In particular,

$$K_1(N) \leqslant K_1 \vee 0, \qquad N \geqslant N_0.$$

Thus, when $E$ is finite, if we replace $Q_u$, $C_u$, $K_1$, $K_2'$ and $K_2''$ by $Q_u^{(N)}$, $C_u^{(N)}$, $\overline{K}_1 := K_1 \vee 0$, $\overline{K}_2' := K_2' \vee 0$ and $\overline{K}_2'' := K_2'' \vee 0$, respectively, and define the corresponding $\overline{K}_2$, $\bar{c}$ and $\overline{K}_3$, then the estimates (21) and (29) are still available. Finally, if we again replace $S$ with $\Lambda_n$, these remarks are also available.

**Lemma 2.5.** Let $S$ be finite. Then for each $t \geqslant 0$ we choose a subsequence $\{N_\ell\}$ such that

$$\lim_{\ell\to\infty} S^{(N_\ell)}(t) f(\eta) = S(t) f(\eta), \qquad f \in \mathscr{L}_b,\ \eta \in \mathscr{E},$$

where $S^{(N)}(t)$ is the semigroup determined by $\Omega^{(N)}$ and $\mathscr{L}_b$ is the set of all bounded $\mathscr{L}$-functions.

*Proof.* Since the $Q$-matrix $Q^{(N)}$ corresponding to $\Omega^{(N)}$ is bounded and

$$Q^{(N)} = \big(q^{(N)}(\eta, \zeta)\big) \text{ converges to } (q(\eta, \zeta)) \text{ pointwise as } N \to \infty,$$

it follows from [2; Lemma 2] that

$$\varliminf_{N\to\infty} P^{(N)}(t,\eta,\zeta)=P(t,\eta,\zeta),\qquad \eta,\ \zeta\in E,\ t\geqslant 0,$$

since $S$ is finite, hence $E=\mathbb{Z}_+^S$ is countable. Using the pointwise convergence and

$$\sum_\zeta P^{(N)}(t,\eta,\zeta)=1=\sum_\zeta P(t,\eta,\zeta),\qquad \eta\in E,$$

we get

$$\lim_{\ell\to\infty}\sum_\zeta \big|P^{(N_\ell)}(t,\eta,\zeta)-P(t,\eta,\zeta)\big|=0,\qquad \eta\in E.$$

This implies Lemma 2.5. $\square$

From now on, we use $S_n(t)$ to denote the semigroup determined by $\Omega_n$. Its state space is $\mathbb{Z}_+^{\Lambda_n}$. We are going to estimate $|S_n(t)f(\eta)-S_m(t)f(\zeta)|$. To this end, we may assume $m\geqslant n$, and extend naturally $S_n(t)$ to be a semigroup with state space $\mathbb{Z}_+^{\Lambda_m}$.

**Proposition 2.6.** Let $S=\Lambda_m$, $m\leqslant n$. Then, for each $t\geqslant 0$, $f\in\mathscr{L}$ and $\eta\in E$, we have

$$\begin{aligned}
&|S_n(t)f(\eta)-S_m(t)f(\eta)|\\
&\leqslant 2L(f)\sum_{u\in\Lambda_m\setminus\Lambda_n}\sum_{k\neq 0}q_u(\eta(u),\eta(u)+k)|k|\alpha(u)\int_0^t \exp\big[(t-s)\big(\overline{K}_2+K(M+1)\big)\big]\mathrm{d}s\\
&\quad+2KL(f)\int_0^t \exp\big[(t-s)\big(\overline{K}_2+K(M+1)\big)\big]\\
&\qquad\times\sum_{x,y,z}\bigg[\eta(z)e^{s\overline{K}_1}+C\beta(z)e^{s\overline{K}_3}\sum_{\ell=0}^\infty\frac{(Ks)^\ell}{\ell!}P_n^{(\ell)}(z,x)|P_n(x,y)-P_m(x,y)|(\alpha(x)+\alpha(y))\bigg]\mathrm{d}s\\
&=:2C(t,f,\eta,m,n). \qquad (31)
\end{aligned}$$

*Proof.* Let $N\geqslant N_0$. Denote by $\mathbb{E}^\eta_{n,N}$ the expectation corresponding to $\{S_n^{(N)}(t)\}$ with initial state $\eta$. Observe

$$\begin{aligned}
\big|\Omega_n^{(N)}f(\eta)-\Omega_m^{(N)}f(\eta)\big|&\leqslant L(f)\sum_{u\in\Lambda_m\setminus\Lambda_n}\sum_{k\neq 0}q_u^{(N)}(\eta(u),\eta(u)+k)|k|\alpha(u)\\
&\quad+KL(f)\sum_{u,v}\eta(u)|P_n(u,v)-P_m(u,v)|(\alpha(u)+\alpha(v)).
\end{aligned}$$

From (21) and (29), it follows that if $N\geqslant\max_{u\in\Lambda_m\setminus\Lambda_n}\eta(u)$, then

$$\begin{aligned}
&\bigg|\int_0^t S_n^{(N)}(s)\big(\Omega_n^{(N)}-\Omega_m^{(N)}\big)S_m^{(N)}(t-s)f(\eta)\mathrm{d}s\bigg|\\
&\leqslant\int_0^t\mathbb{E}_{n,N}\bigg\{L\big(S_m^{(N)}(t-s)f\big)\bigg[\sum_{u\in\Lambda_m\setminus\Lambda_n}\sum_{k\neq 0}q_u^{(N)}(\eta_s(u),\eta_s(u)+k)|k|\alpha(u)\\
&\quad+K\sum_{u,v}\eta_s(u)|P_n(u,v)-P_m(u,v)|(\alpha(u)+\alpha(v))\bigg]\bigg\}\mathrm{d}s\\
&=\int_0^t\mathbb{E}_{n,N}\bigg\{L\big(S_m^{(N)}(t-s)f\big)\bigg[\sum_{u\in\Lambda_m\setminus\Lambda_n}\sum_{k\neq 0}q_u^{(N)}(\eta(u),\eta(u)+k)|k|\alpha(u)\\
&\quad+K\sum_{u,v}\eta_s(u)|P_n(u,v)-P_m(u,v)|(\alpha(u)+\alpha(v))\bigg]\bigg\}\mathrm{d}s\\
&\leqslant C(t,f,\eta,m,n).
\end{aligned}$$

On the other hand, since $\Omega_n^{(N)}$ and $\Omega_m^{(N)}$ are bounded, and so

$$S_n^{(N)}(t)f(\eta) - S_m^{(N)}(t)f(\eta) = \int_0^t S_n^{(N)}(s)\big(\Omega_n^{(N)} - \Omega_m^{(N)}\big)S_m^{(N)}(t-s)f(\eta)\mathrm{d}s.$$

We have

$$\begin{gathered}\big|S_n^{(N)}(t)f(\eta) - S_m^{(N)}(t)f(\eta)\big| \leqslant C(t,f,\eta,m,n),\\ f \in \mathscr{L}_b,\ \eta \in E,\ t \geqslant 0,\ N \geqslant \max_{u\in\Lambda_m\setminus\Lambda_n} \eta(u).\end{gathered}$$

By Lemma 2.5, for each $t \geqslant 0$, we may choose a subsequence $\{N_\ell\}_1^\infty$, such that

$$\begin{aligned}&\lim_{\ell\to\infty} S_n^{(N_\ell)}(t)f(\eta) = S_n(t)f(\eta),\\ &\lim_{\ell\to\infty} S_m^{(N_\ell)}(t)f(\eta) = S_m(t)f(\eta), \qquad f \in \mathscr{L}_b,\ \eta \in E,\end{aligned}$$

and therefore,

$$\begin{gathered}\big|S_n(t)f(\eta) - S_m(t)f(\eta)\big| \leqslant C(t,f,\eta,m,n),\\ f \in \mathscr{L}_b,\ \eta \in E,\ t \geqslant 0.\end{gathered}$$

Now, the estimate (31) follows immediately. □

## §3. Proof of Theorem 1.1

In this section, we will assume that $S$ is infinite, $\{\Lambda_n\}_1^\infty$ is a sequence of finite subsets of $S$, increasing to $S$. First of all, we notice that if we consider $S_n(t)$ as a semigroup on $\mathscr{L}$ with state space $\mathscr{E}$, then the estimates (21), (29) and (31) are all available, and the only change is replacing $M$ by $M+1$.

To construct a semigroup $S(t)$ on $\mathscr{L}$, the main point is the following result:

**Proposition 3.1.** For each $f \in \mathscr{L}$, $\eta \in \mathscr{D}$ and $t \geqslant 0$, $\{S_n(t)f(\eta) : n \geqslant 1\}$ is a Cauchy sequence.

*Proof.* Express

$$C(t,f,\eta,m,n) = I_1 + I_2,$$

where

$$\begin{aligned}I_1 &= L(f)\sum_{u\in\Lambda_m\setminus\Lambda_n}\sum_{k\neq 0} q_u(\eta(u),\eta(u)+k)|k|\alpha(u)\int_0^t \exp\big[(t-s)\big(\overline{K}_2 + K(M+1)\big)\big]\mathrm{d}s\\ &=: \Bigg[\sum_{u\in\Lambda_m\setminus\Lambda_n,\eta(u)=0} q_u(0,k)|k|\alpha(u)\\ &\qquad + \sum_{u\in\Lambda_m\setminus\Lambda_n,\eta(u)\neq 0}\sum_{k\neq 0} q_u(\eta(u),\eta(u)+k)|k|\alpha(u)\Bigg]C(t,f)\\ &= C(t,f)\Bigg[\sum_{u\in\Lambda_m\setminus\Lambda_n}\alpha(u)\beta(u) + \sum_{u\in\Lambda_m\setminus\Lambda_n,\eta(u)\neq 0}\sum_{k\neq 0} q_u(\eta(u),\eta(u)+k)|k|\alpha(u)\Bigg].\end{aligned}$$

Since $\eta \in \mathscr{D}$ and the condition (5), we get

$$I_1 \to 0, \qquad \text{as} \qquad m,n \to \infty.$$

Next, the integrand of $I_2$ is bounded by

$$e^{(t-s)\left(\overline{K}_2+K(M+1)\right)}\sum_{x,z}\left[\eta(z)e^{s\overline{K}_1}+C\beta(z)e^{s\overline{K}_3}\sum_{\ell=0}^{\infty}\frac{(Ks)^\ell}{\ell!}P_n^{(\ell)}(z,x)\right]2(M+1)\alpha(x)$$
$$\leqslant 2(M+1)\left(\|\eta\|e^{s\overline{K}_1}+C\|\beta\|e^{s\overline{K}_3}\right)e^{(t-s)\left(\overline{K}_2+K(M+1)(1+s)\right)}$$

Then from $P_n(x,y)\to P(x,y)$ and the dominated convergence theorem we have

$$I_2\to 0,\qquad \text{as}\qquad m,n\to\infty.$$

The proof of Proposition 3.1 is now completed. □

By Proposition 3.1, we may define

$$S(t)f(\eta)=\lim_{n\to\infty}S_n(t)f(\eta),\qquad f\in\mathscr{L},\ \eta\in\mathscr{D}; \tag{32}$$

since (29) and Remark 2.4, we get

$$|S(t)f(\eta)-S(t)f(\zeta)|\leqslant L(f)\|\eta-\zeta\|e^{t(K_2+K(M+1))},\qquad f\in\mathscr{L},\ \eta,\zeta\in\mathscr{D}. \tag{33}$$

This shows that $S(t)f(\cdot)$ is uniformly continuous on $\mathscr{D}$. On the other hand, $\mathscr{D}$ is dense in $\mathscr{E}$. By an elementary extension theorem, $S(t)f$ can be extended to whole $\mathscr{E}$ as a uniformly continuous function, again denoted by $S(t)f$. Thus, $\mathscr{D}$ in (33) can be replaced by $\mathscr{E}$, and so we get (14).

**Lemma 3.2.** $S(t)$ is a positive operator on $\mathscr{L}$, and it is contraction on $\overline{\mathscr{L}}$. Moreover, for each $f\in\mathscr{L}$ such that $f(\eta)\leqslant\widetilde{C}\|\eta\|$, we have

$$|S(t)f(\eta)|\leqslant\widetilde{C}\|\eta\|e^{t(K_2+K(M+1))}+\widetilde{C}C\|\beta\|e^{t(K_3+K(M+1))}. \tag{34}$$

*Proof.* From

$$\begin{aligned}|S(t)f(\eta)|&\leqslant|S_m(t)f(\eta)|+|S(t)f(\eta)-S_m(t)f(\eta)|\\&\leqslant\sup_\eta|f(\eta)|+|S(t)f(\eta)-S_m(t)f(\eta)|,\end{aligned}$$

it follows that

$$|S(t)f(\eta)|\leqslant\sup_\eta|f(\eta)|,\qquad f\in\mathscr{L}_b,\ \eta\in\mathscr{D}.$$

Because of the uniformly continuity of $S(t)f$ on $\mathscr{E}$, we see that the last inequality holds for all $\eta\in\mathscr{E}$. Similarly, it can be shown that $S(t)$ is positive on $\mathscr{L}$. Now, we need only to check (34) for $f(\eta)=\eta$, but this is a straightforward consequence of (29) and (21). □

We have proved the main part of Theorem 1.1. The remains of the proof are now not hard to complete for the readers who are familiar with the machinery developed by Liggett and Spitzer in [8].

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