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Dirichlet Forms and Related Topics

In Honor of Masatoshi Fukushima's Beiju, IWDFRT 2022, Osaka, Japan, August 22–26

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Contents

Hermitizable, Isospectral Matrices or Differential Operators

Mu-Fa Chen

Abstract This paper reports the study on Hermitizable problem for complex matrix or second order differential operator. That is the existence and construction of a positive measure such that the operator becomes Hermitian on the space of complex square-integrable functions with respect to the measure. In which case, the spectrum are real, and the corresponding isospectral matrix/differntial operators are described. The problems have a deep connection to computational mathematics, stochastics, and quantum mechanics.

Keywords Hermitizable · Matrix · Differential operators · Isospectrum

Mathematics Subject Classification 15A18 · 34L05 · 35P05 · 37A30 · 60J27

According to the different objects: matrix and differential operator, the report is divided into two sections, with emphasis on the first one.

1 Hermitizable, Isospectral Matrices

Let us start at the countable state space $E = \{k \in \mathbb{Z}_+ : 0 \le k < N + 1\}$ $(N \le \infty)$. Consider the tridiagonal matrix T or Q as follows:

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$$\begin{matrix} T \\ Q \end{matrix} = \begin{pmatrix} -c_0 & b_0 & & & 0 \\ a_1 & -c_1 & b_1 & & \\ & a_2 & -c_2 & b_2 & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & a_N & -c_N \end{pmatrix},$$

where for matrix T: the three sequences (a_k), (b_k), (c_k) are complex; and for (birth-death, abbrev. BD-) matrix Q: the subdiagonal sequences (a_k) and (b_k) are positive, and the diagonal one satisfies $c_k = a_k + b_k$ for each $k < N$, except $c_N \geq a_N$ if $N < \infty$. For short, we often write T (or Q) $\sim (a_k, -c_k, b_k)$ to denote the tridiagonal matrix. It is well known that the matrix Q possesses the following property:

$$\mu_n a_n = \mu_{n-1} b_{n-1}, \quad 1 \leq n < N + 1 \tag{1}$$

for a positive sequence $(\mu_k)_{k\in E}$. Actually, property (1) is equivalent to

$$\mu_n = \mu_{n-1} \frac{b_{n-1}}{a_n}, \quad 1 \leq n < N + 1 \;\text{ with initial }\; \mu_0 = 1. \tag{2}$$

In other words, at the present simple situation, one can write down (μ_k) quite easily: starting from $\mu_0 = 1$, and then compute $\{\mu_k\}_{k=1}^N$ step by step (one-step iteration) along the path

$$0 \to 1 \to 2 \to \cdots .$$

At the moment, it is somehow strange to write T and Q together, since they are rather different. For T, three complex sequences are determined by 6 real sequences and Q is mainly determined by two positive sequences, or equivalently, only one real sequence. However, it will be clear later, these two sequences have some special "blood kinship", a fact discovered only three years ago [6, Sect. 3].

Clearly, for Q, property (1) is equivalent to

$$\mu_i a_{ij} = \mu_j a_{ji}, \quad i, j \in E, \tag{3}$$

provided we re-express the matrix Q as $(a_{ij} : i, j \in E)$ since except the symmetric pair (a_n, b_{n-1}) given in (1), for the other i, j, the equality (3) is trivial. However, for general real $A = (a_{ij} : i, j \in E)$, property (3) is certainly not trivial.

Definition 1 A real matrix $A = (a_{ij} : i, j \in E)$ is called *symmetrizable* if there exists a positive measure $(\mu_k : k \in E)$ such that (3) holds.

The meaning of (3) is as follows. Even though A itself is not symmetric, but once it is evoked by a suitable measure (μ_k), the new matrix $(\mu_i a_{ij} : i, j \in E)$ becomes symmetric. Every one knows that the symmetry is very important not only in nature, but also in mathematics. Now how far away is it from symmetric matrix to the symmetrizable one? Consider $N = \infty$ in particular. Then symmetry means that $\mu_k \equiv$ a nonzero positive constant, and so as a measure, μ_k can not be normalized as a probability one. Hence, there is no equilibrium statistical physics since for which, the equilibrium measure should be a Gibbs measure (a probability measure). Next, in this case, the most part of stochastics is not useful since the system should die out.

A systemic symmetrizable theory was presented by Hou and Chen in [13] in Chinese (note that it was too hard to obtain necessary references and so the paper was done without knowing what happened earlier out of China). The English abstract appeared in [14]. Having this tool at hand, our research group was able to go to the equilibrium statistical physics, as shown in [2, Chaps. 7, 11 and Sect. 14.5].

One of the advantage of the symmetric matrix is that it possesses the real spectrum. This is kept for the symmetrizable one. When we go to complex matrix, the symmetric matrix should be replaced by the Hermitian one for keeping the real spectrum. This leads to the following definition.

Definition 2 A complex matrix $A = (a_{ij} : i, j \in E)$ is called *Hermitizable* if there exists a positive measure $(\mu_k : k \in E)$ such that

$$\mu_i a_{ij} = \mu_j \bar{a}_{ji}, \quad i, j \in E, \tag{4}$$

where $\bar{a}$ is the conjugate of a.

Clearly, in parallel to the real case, even though A itself is not Hermitian, but once it is evoked by a suitable measure (μ_k), the new matrix $(\mu_i a_{ij} : i, j \in E)$ becomes Hermitian. Both A and $(\mu_i a_{ij} : i, j \in E)$ have real spectrum.

From (4), we obtain the following simple result.

Lemma 3 *In order the complex $A = (a_{ij})$ to be Hermitizable, the following conditions are necessary.*

- *The diagonal elements $\{a_{ii}\}$ must be real.*
- *Co-zero property: $a_{ij} = 0$ iff $a_{ji} = 0$ for all i, j.*
- *Positive ratio: $\dfrac{\bar{a}_{ij}}{a_{ji}} = \dfrac{a_{ij}}{\bar{a}_{ji}} > 0$ or equivalently, positive product: $a_{ij}a_{ji} > 0$.*

Proof The last assertion of the lemma comes from the following identity:

$$\frac{\alpha}{\bar{\beta}} = \frac{\alpha\beta}{|\beta|^2}, \qquad \beta \neq 0. \quad \square$$

Combining the lemma with the result on BD-matrix, we obtain the following conclusion.

Theorem 4 (Chen [6, Corollary 6]) *The complex T is Hermitizable iff the following two conditions hold simultaneously.*

- *(c_k) is real.*
- *Either $a_{k+1} = 0 = b_k$ or $a_{k+1}b_k > 0$ for each k: $0 \le k < N$.*

Then, we have

$$\mu_0 = 1, \quad \mu_n = \mu_{n-1}\frac{b_{n-1}}{\bar{a}_n} = \mu_0 \prod_{k=1}^{n} \frac{b_{k-1}}{\bar{a}_k}.$$

In practice, we often ignore the part "$a_{k+1} = 0 = b_k$" since otherwise, the matrix T can be separated into two independent blocks.

We now come to the general setup. First, we write $i \to j$ once $a_{ij} \neq 0$. Next, a given path $i_0 \to i_1 \to \cdots \to i_n$ is said to be closed if $i_n = i_0$. A closed one is said to be smallest if it contains no-cross or no round-way closed path. A round-way path means $i_0 \to i_1 \to i_2 \to i_1 \to i_0$ for example. In particular, each closed path for T must be round-way.

Theorem 5 (Chen [6, Theorem 5]) *The complex $A = (a_{ij})$ is Hermitizable iff the following two conditions hold simultaneously.*

- *Co-zero property. For each pair i, j, either $a_{ij} = 0 = a_{ji}$ or $a_{ij}a_{ji} > 0$ (which implies that (a_{kk}) is real).*
- *Circle condition. For each smallest (no-cross-) closed path $i_0 \to i_1 \to \cdots \to i_n = i_0$, the circle condition holds*

$$a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = \bar{a}_{i_n i_{n-1}} \cdots \bar{a}_{i_2 i_1} \bar{a}_{i_1 i_0}.$$

In words, the product of $a_{i_k i_{k+1}}$ along the path equals to the one of product of $\bar{a}_{i_{k+1} i_k}$ along the inversive direction of the path.

Proof One may check that our Hermitizability is equivalent to A being Hermitian on the space $L^2(\mu)$ of square-integrable complex function with the standard inner product

$$(f, g) = \int f \bar{g} \, \mathrm{d}\mu.$$

Hence the Hermitizability seems not new for us. However, the author does not know up to now any book tells us how to find out the measure μ. Hence, our main task is to find such a measure if possible. Here we introduce a very natural proof of Theorem 5, which is published here for the first time.

Next, in view of the construction of μ for BD-matrix Q or T, one can find out the measure step by step along a path. We now fix a path as follows.

$$i_0 \to i_1 \to \cdots \to i_{n-1} \to i_n, \qquad a_{i_k i_{k+1}} \neq 0.$$

Comparing the jumps and their rates for BD-matrix and the present A:

$$\begin{aligned} k-1 \to k: b_{k-1}, \quad & i_{k-1} \to i_k: a_{i_{k-1} i_k}, \\ k \to k-1: \ \bar{a}_k, \quad & i_k \to i_{k-1}: \bar{a}_{i_k i_{k-1}}. \end{aligned}$$

From the iteration for BD-matrix

$$\mu_n = \mu_{n-1} \frac{b_{n-1}}{\bar{a}_n},$$

it follows that for the matrix A along the fixed path above, we should have

$$\mu_{i_n} = \mu_{i_{n-1}} \frac{a_{i_{n-1} i_n}}{\bar{a}_{i_n i_{n-1}}}.$$

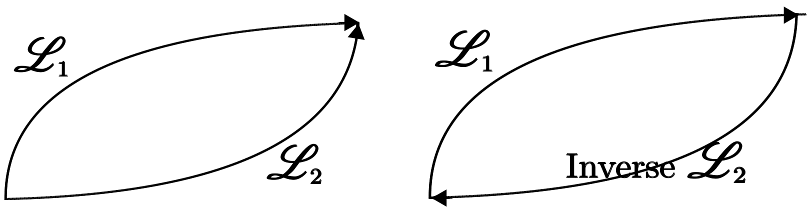
Therefore, we obtain

$$\prod_{k=1}^{n} \frac{a_{i_{k-1} i_k}}{\bar{a}_{i_k i_{k-1}}} = \frac{\mu_{i_n}}{\mu_{i_0}}. \tag{5}$$

Thus, if we fixed i_0 to be a reference point, then we can compute μ_{i_k} $(k = 1, 2, \cdots, n)$ successively by using this formula. The essential point appears now, in the present general situation, there may exist several paths from the same $j_0 = i_0$ to the same $j_m = i_n$, as shown in the left figure below. We have to show that along these two paths, we obtain the same $\mu_{i_n} = \mu_{j_m}$. That is the so-called path-independence. This suggests us to use the conservative field theory in analysis. The path-independence is equivalent to the following conclusion: the work done by the field along each closed path equals zero. This was the main idea we adopted in [13]. To see it explicitly, from (5), it follows that

$$w(\mathscr{L}_1) := \sum_{k=1}^{n} \log \frac{a_{i_{k-1} i_k}}{\bar{a}_{i_k i_{k-1}}} = \log \mu_{i_n} - \log \mu_{i_0}.$$

The left-hand side is the work done by the conservative field along the path $\mathscr{L}_1$: $i_0 \to \cdots \to i_{n-1} \to i_n$, and the right-hand side is the difference of potential of the field at positions i_n and i_0. Clearly, once $i_n = i_0$, the right-hand side equals zero (let call it the conservativeness for a moment).



Left figure: two paths from i_0 to $i_\#$: $\mathscr{L}_1$ and $\mathscr{L}_2$. **Right figure**: combining $\mathscr{L}_1$ and inversive $\mathscr{L}_2$ together, we get a closed path.

For the reader's convenience, we check the equivalence of the path-independence

$$w(\mathscr{L}_1) = w(\mathscr{L}_2)$$

and the conservativeness of the field in terms of the right figure

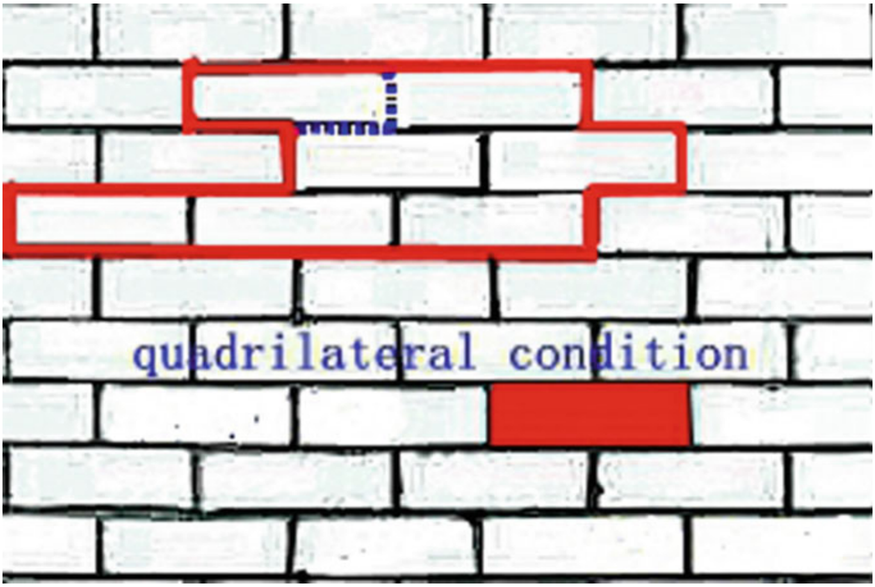
$$w(\mathscr{L}_1) + w(\text{Inverse}\,\mathscr{L}_2) = 0.$$

The conclusion is obvious by using the third assertion of Lemma 3:

$$w(\text{Inverse}\,\mathscr{L}_2) = -w(\mathscr{L}_2).$$

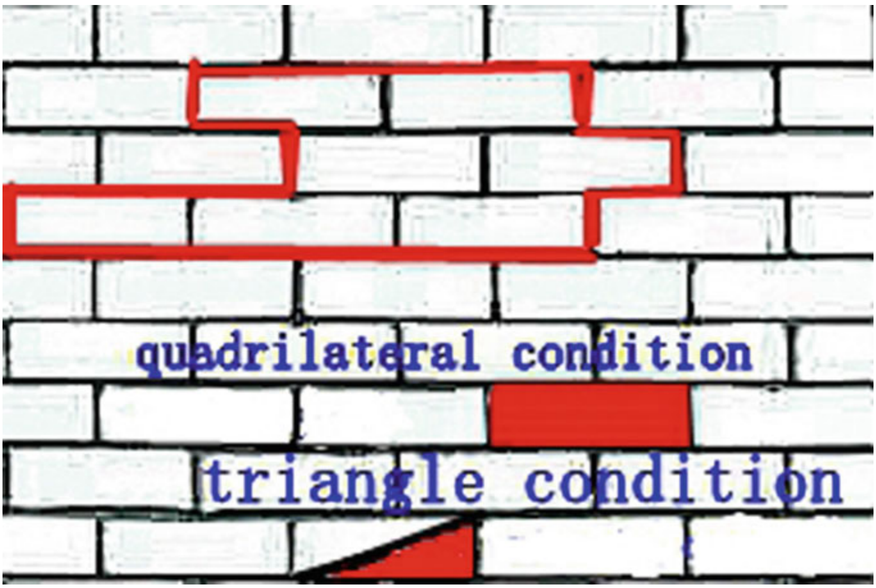
The last property is exactly the circle condition given in the theorem, and so the proof is finished. □

In the special case that A is a transition probability of a finite time-discrete Markov chain, the circle condition was obtained by Kolmogorov [15], as a criterion of the reversibility of the Markov chain. It is also interesting that at the beginning and at the end of [15], the paper by Schrödinger [17] was cited. Moreover, Kolmogorov studied the reversible diffusion in 1937 [16]. These two papers [15, 16] begun the research direction of reversible Markov processes (and also the modern Dirichlet form theory). It also indicates the tight relation between the real symmetrizable operators and equilibrium statistical physics. Nevertheless, the interacting subjects "random fields" and "interacting particle systems" were only born in 1960s. Even though there are some publications along this line, the "Schrödinger diffusion" for instance [1], we are not sure how a distance now to the original aim of Schrödinger who was looking for an equation derived from classical probability, which is as much close as possible to his wave equation in quantum mechanics.



It is regretted that the author had a chance to read [15, 16] only a few years ago when "Selected Works of A. N. Kolmogorov" appeared. Hence, the author did not know anything earlier about Kolmogorov's [15, 16]. There is a Chinese proverb that says "the ignorant are fearless". For this reason, we were brave enough to make a restriction "smallest closed path" instead of "every closed one" in the theorem and then we had gone for much far away, since the total number of the closed paths may be infinite, even not countable. To illustrate the idea, let us consider a random chosen wall above. One sees that there are a lot of closed paths. However, the smallest one is quadrilateral. Hence, one has to check only the "quadrilateral condition". To see this, look at the closed path on the top, and it consists of 7 quadrilaterals. The short path with dash line on the top separates the whole closed path into two smaller ones. To prove that it sufficient to check the "quadrilateral condition" for this model, we use induction. The idea goes as follows. We can make first the union of these two smaller closed paths (choose the clockwise direction for one of the closed path and choose anti-clockwise for the other one). Then remove the round-way path with dash line. Thus, once the work done by the field along each of the smaller closed paths equals zero, then so is the one along the original closed path since the work done by the field along the round-way path equals zero.

However, for the second wall below, the smallest closed path, except the quadrilateral, there is also triangle, so we have the "triangle condition". It is interesting, in [2, Chaps. 7 and 11], we use only these two conditions; and in [2, Sect. 14.5], we use only the triangle condition. The main reason is that for infinite-dimensional objects, their local structures are often regular and simple. Besides, in general we have an algorithm to justify the Hermitizability by computer, refer to [10, Algorithm 1].



We are now arrive at the core part of the paper: describing the spectrum of the Hermitizable matrix, which is also the core part of the so-called matrix mechanics. The next result explains the meaning of "blood kinship" mentioned at the beginning of this section.

Theorem 6 (Chen [6, Corollary 21]) *Up a shift if necessary, each irreducible Hermitizable tridiagonal matrix T is isospectral to a BD-matrix $\widetilde{Q}$ which can be expressed by the known sequences (c_k) and $(a_{k+1}b_k)$.*

The main condition we need for the above result is $c_k \ge |a_k| + |b_k|$ for every $k \in E$. For finite E, the condition is trivial since one may replace (c_k) by a shift $(c_k + m)$ for a large enough constant m. For infinite E, one may require this assumption up to a shift.

We now state the construction of $\widetilde{Q} \sim \left(\tilde{a}, -\tilde{c}_k, \tilde{b}_k\right)$. The essential point is the sequence $\left(\tilde{b}_k\right)$:

$$\tilde{b}_k = c_k - \frac{u_k}{\tilde{b}_{k-1}}, \qquad \tilde{b}_0 = c_0,$$

where $u_k := a_k b_{k-1} > 0$. This is one-step iteration, and we have the explicit expression

$$\tilde{b}_k = c_k - \cfrac{u_k}{c_{k-1} - \cfrac{u_{k-1}}{c_{k-2} - \cfrac{u_{k-2}}{\ddots\, c_2 - \cfrac{u_2}{c_1 - \cfrac{u_1}{c_0}}}}}.$$

Note that here two sequences (c_k) and (u_k) are explicit known. Having $\left(\tilde{b}_k\right)$ at hand, it is easy to write down $\tilde{a}_k = \tilde{c}_k - \tilde{b}_k$ with $\tilde{c}_k = c_k$ for $k < N$, and $\tilde{a}_N = u_N/\tilde{b}_N$ if $N < \infty$. The solution of $(\tilde{a}_k)$ and $(\tilde{c}_k)$ are automatic so that $\widetilde{Q}$ becomes a BD-matrix.

The resulting matrix $\widetilde{Q}$ looks very simple, but it contains a deep intrinsic feature. For instance, the reason is not obvious why the sequences $\left(\tilde{b}_k\right)$ and $(\tilde{a}_k)$ are positive even though so are (c_k) and (u_k). With simple description but deep intrinsic feature is indeed a characteristic of a good mathematical result.

To see the importance of the above theorem, let compare the difference of the principal eigenvector of these two matrices. First, for BD-matrix with four different boundaries, the principal eigenvectors are all monotone, except in one case, it is concave. This enables us to obtain a quite satisfactory theory of the principal eigenvalues (refer to [4]). However, since the Hermitizable T has real spectrum, form the eigenequation

$$Tg = \lambda g,$$

one sees immediately, the eigenvector g must be complex, too far away to be monotone. Thus, the principal eigenvectors of these two operators are essentially different.

It shows that we now have a new spectral theory for the Hermitizable tridiagonal matrices.

Because the intuition is not so clear why Theorem 6 should be true, two alternative proofs are presented in [7].

Theorem 7 (Chen [6, Theorem 24]) *The spectrum of a finite Hermitizable matrix A is equal to the union of the spectrums of m BD-matrices, where m is the largest multiplicity of eigenvalues of A.*

Refer to ([10, Proofs in §4]) for details. The proof is based on Theorem 6 and the "Householder transformation" which is one of the 10 top algorithms in the twentieth century. The restriction to the finite matrix is due to the use of the transformation. The number m is newly added here which comes from the fact that the eigenvalues of BD-matrices must be distinct and simple, as illustrated by [10, Example 9].

Theorem 7 provides us a new architecture for the study on matrix mechanics (and then for quantum mechanics) since we have a unified setup (BD-matrix) to describe its spectrum. This leads clearly to a new spectrum theory, as illustrated by [7] for tridiagonal matrix and by [11] for one-dimensional diffusions. It also leads to some new algorithms for computational mathematics, as illustrated by [9, 10].

2 Hermitizable, Isospectral Differential Operators

Two Approaches for Studying the Schrödinger Operator

(1) The most popular approach to study the Schrödinger operator

$$L = \frac{1}{2}\Delta + V$$

is the Feynman-Kac semigroup $\{T_t\}_{t\geq 0}$:

$$T_t f(x) = \mathbb{E}_x\left\{ f(w_t)\exp\left[\int_0^t V(w_s)\mathrm{d}s\right]\right\},$$

where (w_t) is the standard Brownian motion. This is often an unbounded semigroup. The Schrödinger operator was born for quantum mechanics, and it is 95 years older this year. In the past hundred years or so, there are a huge number of publications devoted to the Schrödinger operator. However, for the discrete spectrum which is the most important problem in quantum mechanics, the useful results are still very limited as far as we know. In particular, even in dimension one, we have not seen the results which are comparable with [5].

(2) As in the first section, this paper introduces a new method to study the spectrum of Schrödinger operator. That is, replacing the operator L above by

$$\tilde{L} = \frac{1}{2}\Delta + \tilde{b}^h \nabla,$$

where h is a harmonic function: $Lh = 0$, $h \neq 0$ (a.e.). Then, the operator L on $L^2(\mathrm{d}x)$ is isospectral to the operator $\widetilde{L}$ on $L^2(\tilde{\mu}) := L^2(|h|^2\mathrm{d}x)$.

We now consider a general operator. Let $a_{ij}, b_i, c : \mathbb{R}^d \to \mathbb{C}$, $V : \mathbb{R}^d \to \mathbb{R}$, and set $a = (a_{ij})_{i,j=1}^d$, $b = (b_i)_{i=1}^d$. Define $\mathrm{d}\mu = e^V \mathrm{d}x$ and

$$L = \nabla(a\nabla) + b \cdot \nabla - c.$$

Here is the result on the Hermitizability. Denote by a^H the transpose (a^*) and conjugate ($\bar{a}$) of the matrix a.

Theorem 8 (Chen and Li [11]) *Under the Dirichlet boundary condition, the operator L is Hermitizable with respect to μ iff $a^H = a$ and*

$$\begin{aligned} \mathrm{Re\,b} &= (\mathrm{Re\,a})(\nabla \mathrm{V}), \\ 2\,\mathrm{Im\,c} &= -\big((\nabla V)^* + \nabla^*\big)\big((\mathrm{Im\,a})(\nabla \mathrm{V}) + \mathrm{Im\,b}\big). \end{aligned}$$

Recall that a key point in the isospectral transform of T and $\widetilde{Q}$ is that the resulting matrix $\widetilde{Q}$ obeys the condition $\tilde{c}_k = \tilde{a}_k + \tilde{b}_k$ for each $k < N$, and there is no killing/potential term at the diagonal (maybe except only one at the endpoint if $N < \infty$). In the next result, we also remove the potential term c in L. Since the isospectal property is described by using the quadratic forms, we do not require much of the regularity of h and Lh in the next result.

Theorem 9 (Chen and Li [11]) *Denote by $\mathscr{D}(L)$ the domain of L on $L^2(\mu)$ and let h be harmonic: $Lh = 0$, $h \neq 0$ (a.e.). Then $(L, \mathscr{D}(L))$ is isospectral to the operator $\big(\widetilde{L},\ \mathscr{D}(\widetilde{L})\big)$:*

$$\begin{cases} \widetilde{L} = \nabla(a\nabla) + \tilde{b} \cdot \nabla, \\ \mathscr{D}(\widetilde{L}) = \big\{\tilde{f} \in L^2(\tilde{\mu}) : \tilde{f}h \in \mathscr{D}(L)\big\}; \end{cases}$$

where

$$\tilde{b} = b + 2\,\mathrm{Re}(a)\,\mathbb{1}_{[h\neq 0]}\frac{\nabla h}{h}, \qquad \tilde{\mu} := |h|^2\mu.$$

The discrete spectrum for one-dimensional elliptic differential operator is also illustrated in [11]. Certainly, much of the research work should be done in the near future. For instance, Hermitizable operator is clearly the Hermitian operator on the complex space $L^2(\mu)$. It naturally corresponds to a Dirichlet form. Hence there should be a complex process corresponding to the operator. It seems that this is still a quite open area, except a few of papers, Fukushima and Okada [12] for instance.

In conclusion, the paper [13] published 42 years ago opened a door for us to go to the equilibruim/nonequilibrium statistical physics (cf. [2, 3]); the paper [6] that appeared 3 years ago enables us to go to quantum mechanics. The motivation of the present study from quantum mechanics was presented in details in [8] but omitted here.

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