Ten Lectures on Ergodic Convergence Rates of Markov Processes

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Preface

This file is a collection of ten lectures of pdf-files presented at five universities and four conferences in Japan during my two visits in 2002. The first one is an overview of the second to the seventh lectures. Mainly, we study several different types of convergence or inequalities, by using three mathematical tools: a probabilistic tool—the coupling method, a generalized Cheeger's method originated in Riemannian Geometry, and an approach which comes from Potential Theory and Harmonic Analysis. A diagram of nine types of ergodicity is presented in Lecture 8. The explicit criteria for different types of convergence and the explicit estimates of the convergence rates are given in Lectures 5 and 6. The topics of the last two lectures are different but closely related. Lecture 9 provides the resource of the problems and illustrates some applications. In the last lecture, one can see an interesting application of the first eigenvalue, its eigenfunctions and an ergodic theorem to a stochastic model of economy. Some open problems are also included. Most of the materials used in the lectures can be found from my homepage.

The author is grateful for the kind invitations, financial supports and the warm hospitality made by many Japanese probabilists and their universities.

To them, the file is dedicated with deep acknowledgement. With an apology of missing many names here, a few of them is listed below:

- Professors I. Shigekawa, Y. Takahashi, T. Kumagai and N. Yosida at Kyoto University.
- Professors M. Fukushima, S. Kotani and S. Aida at Osaka University.
- Professors H. Osada, S. Liang at Nagoya University.
- Professors T. Funaki and S. Kusuoka at Tokyo University.

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Mu-Fa Chen, December 20, 2002

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Ergodic Convergence Rates of Markov Processes —Eigenvalues, Inequalities and Ergodic Theory

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Contents

- Introduction.
- Variational formulas of the first eigenvalue by couplings.
- Estimates of inequalities' constant by generalized Cheeger's method.
- A diagram of nine types of ergodicity and a table of explicit criteria in dimension one.

I. Introduction

1.1 Definition. The first (non-trivial) eigenvalue:

$$Q = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$a_i > 0, \; b_i > 0.$$

$Q\mathbb{1} = 0$. Trivial eigenvalue: $\lambda_0 = 0$.

Question: Next eigenvalue of $-Q$: $\lambda_1 = ?$

Elliptic operator in $\mathbb{R}^d$; Laplacian on Riemannian manifolds. Importance: leading term.

1.2 Motivation (Applications)

(1) Phase transitions

$$L = L_\beta,\ P_t = e^{tL}$$

$$\|P_t - \pi\|_{2\to 2} \le e^{-\lambda_1 t}$$

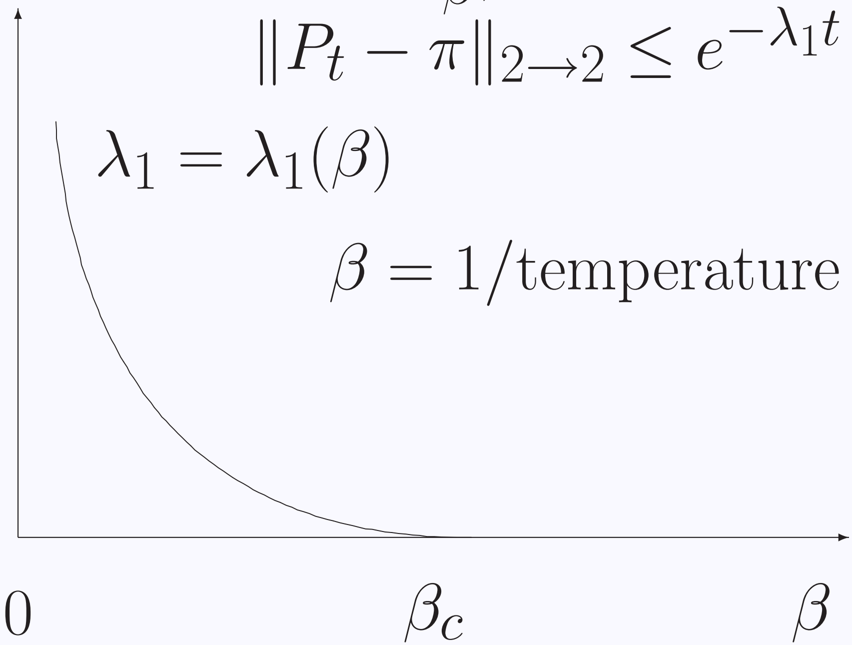

Figure 1: Phase transition and λ_1

(2) Random algorithm: Markov Chains Monte Carlo

NP$\Longrightarrow$P. S. Smale (80's). A. Sinclair (93)

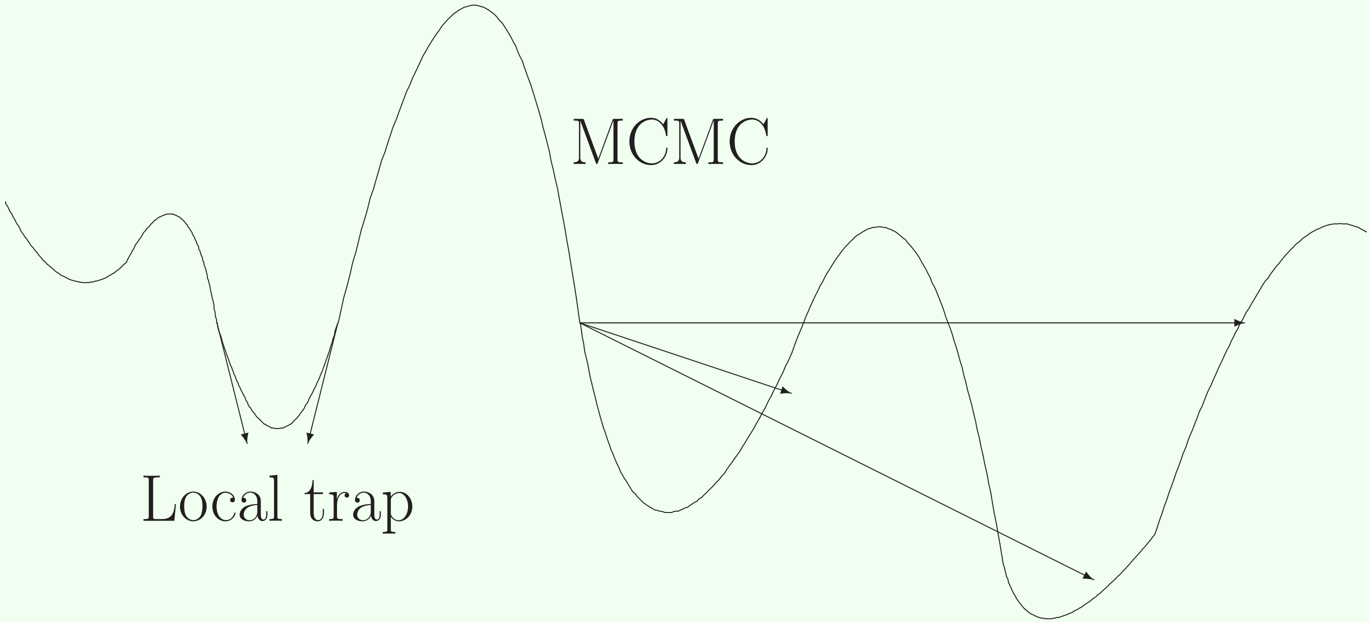

Figure 2: Random algorithm and λ_1

Travelling Salesman Problem:

Find shortest closed path (without loop) among 144 cities in China.

For computer with speed 10^9 path/second:

$$\frac{143!}{10^9 \times 365 \times 24 \times 60 \times 60} \approx 10^{231} \text{ (years)}.$$

Typical NP-problem!

MCMC: 30421 km. Best known: 30380 km.

- Markov chain.
- Stay at lower place with bigger probability. Gibbs principle.
- Algorithm with bigger λ_1 is more effective!

1.3 Difficulties

Example 1: Trivial case(two points). Two parameters.

$$\begin{pmatrix} -b & b \\ a & -a \end{pmatrix}, \quad \lambda_1 = a + b.$$

λ_1 is increasing in each of the parameters!

Example 2: Three points. Four parameters.

$$\begin{pmatrix} -b_0 & b_0 & 0 \\ a_1 & -(a_1+b_1) & b_1 \\ 0 & a_2 & -a_2 \end{pmatrix},$$

$$\lambda_1 = 2^{-1}\Big[a_1 + a_2 + b_0 + b_1 - \sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1 b_1}\Big].$$

Example 3: Four points.

Six parameters: $b_0, b_1, b_2, a_1, a_2, a_3$.

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3}\,\left(3\,B - D^2\right)}{3\,C},$$

where

$$D = a_1 + a_2 + a_3 + b_0 + b_1 + b_2,$$

$$B = a_3\,b_0 + a_2\,\left(a_3 + b_0\right) + a_3\,b_1 + b_0\,b_1 + b_0\,b_2 \\ + b_1\,b_2 + a_1\,\left(a_2 + a_3 + b_2\right),$$

$$C = \left(A + \sqrt{4\left(3\,B - D^2\right)^3 + A^2}\right)^{1/3},$$

$$A = -2a_1^3 - 2a_2^3 - 2a_3^3 + 3a_3^2 b_0 + 3a_3 b_0^2 - 2b_0^3 + 3a_3^2 b_1 - 12a_3 b_0 b_1 + 3b_0^2 b_1 + 3a_3 b_1^2 + 3b_0 b_1^2 - 2b_1^3 - 6a_3^2 b_2 + 6a_3 b_0 b_2 + 3b_0^2 b_2 + 6a_3 b_1 b_2 - 12b_0 b_1 b_2 + 3b_1^2 b_2 - 6a_3 b_2^2 + 3b_0 b_2^2 + 3b_1 b_2^2 - 2b_2^3 + 3a_1^2 (a_2 + a_3 - 2b_0 - 2b_1 + b_2) + 3a_2^2 [a_3 + b_0 - 2(b_1 + b_2)] + 3a_2 \left[a_3^2 + b_0^2 - 2b_1^2 - b_1 b_2 - 2b_2^2 - a_3(4b_0 - 2b_1 + b_2) + 2b_0(b_1 + b_2)\right] + 3a_1 \left[a_2^2 + a_3^2 - 2b_0^2 - b_0 b_1 - 2b_1^2 - a_2(4a_3 - 2b_0 + b_1 - 2b_2) + 2b_0 b_2 + 2b_1 b_2 + b_2^2 + 2a_3(b_0 + b_1 + b_2)\right].$$

The role of each parameter is completely mazed!

Not solvable when space has more than five points!

Conclusion: Impossible to compute λ_1 explicitly!

Perturbation of eigenvalues

Example: Infinite tri-diagonal matrix

(Birth-death processes).

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$		
$i+1$	$2i+\left(4+\sqrt{2}\right)$		

Perturbation of eigenvalues

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$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	
$i+1$	$2i+\left(4+\sqrt{2}\right)$	3	

Perturbation of eigenvalues

Example: Infinite tri-diagonal matrix
(Birth-death processes).

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	2
$i+1$	$2i+\left(4+\sqrt{2}\right)$	3	3

Sensitive. In general, it is too hard to estimate λ_1!

Ⅱ. New variational formulas of the first eigenvalue

2.1 Story of the study on λ_1 in geometry

(M, g): compact Riemannian manifold.

Discrete spectrum: $0 = \lambda_0 < \lambda_1 < \cdots$.

- g: Riemannian metric.
- d: dimension. $\mathbb{S}^d$: $D = \pi$, $\text{Ric} = d-1$.
- D: diameter.
- $\text{Ricci}_M \geqslant K g$ for some $K \in \mathbb{R}$.

Idea: Use geometric quantities d, D and K to estimate λ_k's of Laplacian Δ.

Five books!

- Chavel, I. (1984): Eigenvalues in Riemannian Geometry, Academic Press
- Bérard, P. H. (1986): Spectral Geometry: Direct and Inverse Problem, LNM. vol. 1207, Springer-Verlag. Including 2000 references.
- Schoen, R. and Yau, S. T. (1988): Differential Geometry (In Chinese), Science Press, Beijing, China
- Li, P. (1993): Lecture Notes on Geometric Analysis, Seoul National U., Korea
- Ma, C. Y. (1993): The Spectrum of Riemannian Manifolds (In Chinese), Press of Nanjing U., Nanjing

Ten of the most beautiful lower bounds:

Author(s)	Lower bound: $K \geqslant 0$
A. Lichnerowicz (1958)	$\dfrac{d}{d-1}K$
P. H. Bérard, G. Besson & S. Gallot (1985)	$d\left\{\dfrac{\int_0^{\pi/2}\cos^{d-1}t\,\mathrm{d}t}{\int_0^{D/2}\cos^{d-1}t\,\mathrm{d}t}\right\}^{2/d}$ $K=d-1$
P. Li & S. T. Yau (1980)	$\dfrac{\pi^2}{2\,D^2}$
J. Q. Zhong & H. C. Yang (1984)	$\dfrac{\pi^2}{D^2}$
D. G. Yang (1999)	$\dfrac{\pi^2}{D^2}+\dfrac{K}{4}$

Author(s)	Lower bound: $K \leqslant 0$
P. Li & S. T. Yau (1980)	$\dfrac{1}{D^2(d-1)\exp\left[1+\sqrt{1+16\alpha^2}\right]}$
K. R. Cai (1991)	$\dfrac{\pi^2}{D^2} + K$
D. Zhao (1999)	$\dfrac{\pi^2}{D^2} + 0.52K$
H. C. Yang (1989) & F. Jia (1991)	$\dfrac{\pi^2}{D^2}e^{-\alpha}, \quad \text{if } d \geqslant 5$
H. C. Yang (1989) & F. Jia (1991)	$\dfrac{\pi^2}{2\,D^2}e^{-\alpha'}, \quad \text{if } 2 \leqslant d \leqslant 4$

$$\alpha = D\sqrt{|K|(d-1)}/2,$$
$$\alpha' = D\sqrt{|K|((d-1)\vee 2)}/2.$$

2.2 New variational formulas

Theorem [C. & F. Y. Wang, 1997].

$$\lambda_1 \geqslant \sup_{f\in\mathscr{F}} \inf_{r\in(0,D)} \frac{4f(r)}{\int_0^r C(s)^{-1}\mathrm{d}s \int_s^D C(u)f(u)\mathrm{d}u}$$

Two notations:

$$C(r) = \cosh^{d-1}\left[\frac{r}{2}\sqrt{\frac{-K}{d-1}}\right], \quad r \in (0, D).$$

$$\mathscr{F} = \{f \in C[0, D] : f > 0 \text{ on } (0, D)\}.$$

Classical variational formula:

$$\lambda_1 = \inf\{\textstyle\int_M \|\nabla f\|^2 : f \in C^1, \pi(f) = 0, \pi(f^2) = 1\}$$

Goes back to Lord S. J. W. Rayleigh(1877) or E. Fischer (1905). Gen. R. Courant (1924).

Elementary functions: $1, \sin(\alpha r), \cosh^{1-d}(\alpha r)\sin(\beta r)$,

$$\alpha = D\sqrt{|K|/(d-1)}/2, \quad \beta = \frac{\pi}{2D}.$$

Corollary 1 [C. & F. Y. Wang, 1997].

$$\lambda_1 \geqslant \frac{dK}{d-1}\left\{1-\cos^d\left[\frac{D}{2}\sqrt{\frac{K}{d-1}}\right]\right\}^{-1}, \quad d>1, K\geqslant 0.$$

$$\lambda_1 \geqslant \frac{\pi^2}{D^2}\sqrt{1-\frac{2D^2K}{\pi^4}\cosh^{1-d}\left[\frac{D}{2}\sqrt{\frac{-K}{d-1}}\right]},$$

$$d>1, \quad K\leqslant 0.$$

Corollary 2 [C., E. Scacciatelli and L. Yao, 2002].

$$\lambda_1 \geqslant \frac{\pi^2}{D^2}+\frac{K}{2}, \qquad K\in\mathbb{R}.$$

Representative test function:

$$f(r) = \left(\int_0^r C(s)^{-1}\mathrm{d}s\right)^{\gamma}, \quad \gamma = \frac{1}{2}, 1.$$

$$C(s) = \cosh^{d-1}\left[\frac{s}{2}\sqrt{\frac{-K}{d-1}}\right].$$

$$\delta = \sup_{r\in(0,D)} \left(\int_0^r C(s)^{-1}\mathrm{d}s\right)\left(\int_r^D C(s)\mathrm{d}s\right).$$

Corollary 3 [C., 2000]. $\lambda_1 \geqslant \xi_1$.

$$4\delta^{-1} \geqslant (\delta_n')^{-1} \geqslant \xi_1 \geqslant \delta_n^{-1} \geqslant \delta^{-1},$$

Explicit $(\delta_n')^{-1}\downarrow, \quad \delta_n^{-1}\uparrow.$

Compact M with convex ∂M. New, Lichnerowicz.

2.3 "Proof": Estimation of λ_1

Step 1. g: eigenfunction: $Lg = -\lambda_1 g$, $g \neq$const.
From semigroups theory,

$$\frac{d}{dt} T_t g(x) = T_t L g(x) = -\lambda_1 T_t g(x).$$

ODE. $\quad T_t g(x) = g(x) e^{-\lambda_1 t}. \qquad (1)$

Step 2. Compact space. g Lipschitz w.r.t. ρ: c_g.

Key condition: $\widetilde{T}_t \rho(x,y) \leqslant \rho(x,y) e^{-\alpha t}. \qquad (2)$

$$\begin{aligned} e^{-\lambda_1 t} |g(x) - g(y)| &\leqslant \widetilde{T}_t \big|g(x) - g(y)\big| && \text{(by (1))} \\ &\leqslant c_g \widetilde{T}_t \rho(x,y) \\ &\leqslant c_g \rho(x,y) e^{-\alpha t} && \text{(by (2))} \end{aligned}$$

for all t. Hence $\lambda_1 \geqslant \alpha$. General!

$$(2) \iff \widetilde{L} \rho(x,y) \leqslant -\alpha \rho(x,y).$$

Two key points

- “Good” coupling: Classification of couplings. ρ-optimal couplings. “optimal mass transportation”.
- “Good” distance: Ord.$\Rightarrow$ none. Reduce to dim. one.

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \quad C(x) = \int_0^x \frac{b}{a}$$

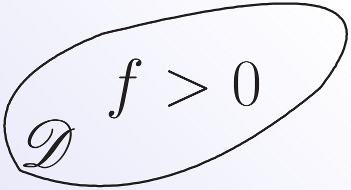
$$f(x) \in C[0, D), \quad f|_{(0,D)} > 0$$

$$g(x) = \int_0^x e^{-C(y)}\mathrm{d}y \int_y^D \frac{fe^C}{a} \uparrow\uparrow$$

$$\rho_f(x, y) = |g(x) - g(y)|.$$

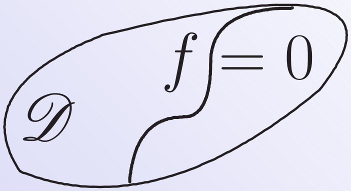
Intrinsic reason of success

Interaction of Prob Theory & Analysis, Geometry.



Dirichlet: $f(\partial\mathscr{D}) = 0$, 99%

$\Longleftrightarrow$ Maximum principle.



Neumann: $\partial f/\partial \mathbf{n} = 0$, $\int f \mathrm{d}\pi = 0$.

Surface $\{f = 0\}$ depends on L, $\mathscr{D}$

Coupling method: $\rho(X_t, Y_t)$, degenerated at T.

Need only $\{t < T\}$. Do not care about position X_T.

2.4 Tri-diagonal matrix (birth-death process)

Notations:

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geqslant 1.$$

$$Z = \sum_i \mu_i < \infty, \qquad \pi_i = \frac{\mu_i}{Z}.$$

$$\mathscr{F} = \{f : f_0 = 0,\ f \text{ is strictly increasing}\},$$

$$\mathscr{F}' = \text{A slight modification of } \mathscr{F}.$$

Theorem [C., 1996, 2000, 2001].

- **Dual variational formulas.** Write $\bar f = f - \pi(f)$.

  $$\lambda_1 = \sup_{f\in\mathscr{F}} \inf_{i\geqslant 0} \mu_i b_i (f_{i+1}-f_i)\Big/\sum\nolimits_{j\geqslant i+1}\mu_j \bar f_j.$$

  $$\lambda_1 = \inf_{f\in\mathscr{F}'} \sup_{i\geqslant 1} \mu_i b_i (f_{i+1}-f_i)\Big/\sum\nolimits_{j\geqslant i+1}\mu_j \bar f_j.$$

- **Explicit estimates.** $Z\delta^{-1} \geqslant \lambda_1 \geqslant (4\delta)^{-1}$, where $\delta = \sup_{i\geqslant 1}\sum_{j\leqslant i-1}(\mu_j b_j)^{-1}\sum_{j\geqslant i}\mu_j$.

- **Approximation procedure.** $\exists$ explicit η_n', η_n'' such that $\quad {\eta_n'}^{-1} \geqslant \lambda_1 \geqslant {\eta_n''}^{-1} \geqslant (4\delta)^{-1}$.

Ⅲ. Basic inequalities and new forms of Cheeger's constants

3.1 Basic inequalities

$(E, \mathscr{E}, \pi)$: prob. space, $L^p(\pi)$, $\|\cdot\|_p$, $\|\cdot\| = \|\cdot\|_2$.

Dirichlet form $(D, \mathscr{D}(D))$ on $L^2(\pi)$.

Poincaré inequality : $\text{Var}(f) \leqslant CD(f),\ C = \lambda_1^{-1}$.

Nash inequality : $\text{Var}(f) \leqslant CD(f)^{1/p}\|f\|_1^{2/q}$

J. Nash (1958) $\qquad 1/p + 1/q = 1$

Logarithmic Sobolev inequality [L. Gross, 1976] :

$$\int_E f^2 \log\left(f^2/\|f\|^2\right) d\pi \leqslant CD(f).$$

3.2 Integral operator.

Symmetric form:

$$D(f) = \frac{1}{2}\int_{E\times E} J(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2,$$

$$\mathscr{D}(D) = \{f \in L^2(\pi) : D(f) < \infty\},$$

$J \geqslant 0$: symmetric, no charge on $\{(x, x) : x \in E\}$.

$$\lambda_1 := \inf\{D(f) : \pi(f) = 0, \|f\| = 1\}.$$

Theorem [G. F. Lawler & A. D. Sokal, 1988] :

$$J(\mathrm{d}x, E)/\mathrm{d}\pi \leqslant M, \quad \lambda_1 \geqslant \frac{k^2}{2M}.$$

Six books:

- Chen, M. F. (1992), From Markov Chains to Non-Equilibrium Particle Systems, World Scientific, Singapore
- Sinclair, A. (1993), Algorithms for Random Generation and Counting: A Markov Chain Approach, Birkhäuser
- Colin de Verdière, Y. (1998), Spectres de Graphes, Publ. Soc. Math. France
- Chung, F. R. K. (1997), Spectral Graph Theory, CBMS, **92**, AMS, Providence, Rhode Island L. (1997),

- Saloff-Coste, L. (1997), Lectures on finite Markov chains, LNM **1665**, 301–413, Springer-Verlag
- Aldous, D. G. & Fill, J. A. (1994–), Reversible Markov Chains and Random Walks on Graphs

3.3 New results.

$$J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y) = I_{\{r(x,y)^\alpha > 0\}} \frac{J(\mathrm{d}x, \mathrm{d}y)}{r(x, y)^\alpha}$$

$r \geqslant 0$, symmetric. $\alpha \in [0, 1]$

$J^{(1)}(\mathrm{d}x, E)/\pi(\mathrm{d}x) \leqslant 1, \quad \pi$-a.e.

New forms of Cheeger's constants

Inequalities	Constant $k^{(\alpha)}$
Poincaré (C.&W.)	$\inf_{\pi(A)\in(0,1)} \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\wedge\pi(A^c)}$
Nash (C.)	$\inf_{\pi(A)\in(0,1)} \frac{J^{(\alpha)}(A\times A^c)}{[\pi(A)\wedge\pi(A^c)]^{(2q-3)/(2q-2)}}$
LogS (Wang)	$\lim_{r\to 0}\inf_{\pi(A)\in(0,r]} \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\sqrt{\log[e+\pi(A)^{-1}]}}$
(C.)	$\lim_{\delta\to\infty}\inf_{\pi(A)>0} \frac{J^{(\alpha)}(A\times A^c)+\delta\pi(A)}{\pi(A)\sqrt{1-\log\pi(A)}}$

Theorem.
$k^{(1/2)} > 0 \Longrightarrow$ corresponding inequality holds.

Four papers:

C. and Wang (1998),
C. (1999, 2000),
Wang (2001).

Estimates $\lambda_1 \geqslant \dfrac{{k^{(1/2)}}^2}{1+\sqrt{1-{k^{(1)}}^2}}$. Can be sharp!

M. Fukushima & T. Uemura (2002):
Criterion of Nash ineq. by isoperimetric constant.

IV. New picture of ergodic theory and explicit criteria

4.1 Importance of the inequalities

$(D, \mathscr{D}(D)) \longrightarrow$ semigroup (P_t): $P_t = e^{tL}$

Theorem [T.M.Liggett(89), L.Gross(76), C.(99)]

- Poincaré ineq.$\iff \text{Var}(P_t f) \leqslant \text{Var}(f) e^{-2\lambda_1 t}$.
- LogS $\Longrightarrow$ exponential convergence in entropy:
$$\text{Ent}(P_t f) \leqslant \text{Ent}(f) e^{-2\sigma t},$$
where $\text{Ent}(f) = \pi(f \log f) - \pi(f) \log \|f\|_1$.
- Nash ineq. $\iff \text{Var}(P_t f) \leqslant C\|f\|_1^2 / t^{q-1}$.

Nash inequality weakest?

4.2 Three traditional types of ergodicity.

$$\|\mu - \nu\|_{\text{Var}} = 2\sup_{A\in\mathscr{E}} |\mu(A) - \nu(A)|$$

Ordinary erg. : $\lim_{t\to\infty} \|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$

Exp. erg. : $\lim_{t\to\infty} e^{\alpha t}\|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$

Strong erg. : $\lim_{t\to\infty} \sup_x \|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$

$$\Longleftrightarrow \lim_{t\to\infty} e^{\beta t} \sup_x \|P_t(x,\cdot) - \pi\|_{\text{Var}} = 0$$

Classical Theorem :

Strong ergodicity $\Longrightarrow$ Exp. erg. $\Longrightarrow$ Ordinary erg.

LogS $\Longrightarrow$ Exp. conv. in entropy $\Longrightarrow$ Poincaré

4.3 New picture of ergodic theory

Theorem. For rever. Markov proc. with densities,

Nash inequality

$\swarrow\!\!\swarrow$ $\qquad\qquad$ $\searrow\!\!\searrow$

LogS inequality $\qquad$ L^1-exp. convergence

$\Downarrow$ $\qquad\qquad$ $\|$

Exp. conv. in entropy $\qquad$ π-a.s. Strong ergodicity

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

Poincaré inequality $\iff$ π-a.s. Exp. ergodicity

$\Downarrow$

L^2-algebraic ergodicity

$\Downarrow$

Ordinary ergodicity

4.4 Explicit criteria for several types of ergodicity

Birth-death processes

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n}, \; n \geqslant 1;$$
$$\mu[i,k] = \sum\nolimits_{i \leqslant j \leqslant k} \mu_j.$$

Theorem. Ten criteria for birth-death processes are listed in the following table [C., 2001].

Property	Criterion
Uniq.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[0,n]=\infty \quad (*)$
Recur.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}=\infty$
Erg.	$(*)$ & $\mu[0,\infty)<\infty$
Exp. erg. Poincaré	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Dis. sp.	$(*)\&\lim_{n\to\infty}\sup_{k\geqslant n+1}\mu[k,\infty)\sum_{j=n}^{k-1}\frac{1}{\mu_j b_j}=0$
LogS	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)\log[\mu[n,\infty)^{-1}]\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Str. erg. L^1-exp.	$(*)\&\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[n+1,\infty)=\sum_{n\geqslant 1}\mu_n\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Nash	$(*)\&\sup_{n\geqslant 1}\mu[n,\infty)^{(q-2)/(q-1)}\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty(\varepsilon)$

Contributors:

- Bobkov, S. G. and Götze, F. (1999a, b)
- C. (1991, 1996, 2000, 2001)
- Miclo, L. (1999, 2000)
- Mao, Y. H. (2001, 2002a, b)
- Wang, F. Y. (2001)
- Zhang, H. Z., Lin, X. and Hou, Z. T. (2000)
- Zhang, Y. H. (2001)
- ......

Optimal Markovian Couplings

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Contents

1. Couplings and Markovian couplings.

Couplings.

$(E_k, \mathscr{E}_k)$ $(k = 1, 2)$: meas. spaces.

Given prob. meas. μ_1, μ_2, a prob. meas. $\widetilde{\mu}$ on $(E_1 \times E_2, \mathscr{E}_1 \times \mathscr{E}_2)$ is called a coupling of μ_1 and μ_2 if following marginality holds:

$$\begin{aligned} \widetilde{\mu}(A_1 \times E_2) &= \mu_1(A_1) \\ \widetilde{\mu}(E_1 \times A_2) &= \mu_2(A_2), \\ A_k \in \mathscr{E}_k, &\quad k = 1, 2. \end{aligned} \qquad \text{(M)}$$

Example 1. **Independent coupling $\tilde{\mu}_0$:**

$\tilde{\mu}_0 = \mu_1 \times \mu_2$.

Application. $\mu_k = \mu$ on $\mathbb{R}$. FKG-inequality:

$$\int fg \mathrm{d}\mu \geqslant \int f \mathrm{d}\mu \int g \mathrm{d}\mu, \quad f, g \in \mathscr{M}.$$

Example 1. **Independent coupling $\tilde{\mu}_0$:**

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$$\int fg\mathrm{d}\mu \geqslant \int f\mathrm{d}\mu \int g\mathrm{d}\mu, \quad f, g \in \mathscr{M}.$$

$$\iint \tilde{\mu}_0(\mathrm{d}x, \mathrm{d}y)[f(x) - f(y)][g(x) - g(y)] \geqslant 0.$$

For diffusions in $\mathbb{R}^d$: complete (C. & Wang, 1993).

For Markov chains: criterion is open.

Example 2. $E_k = E$. Basic coupling $\tilde{\mu}_b$:

$$\tilde{\mu}_b(\mathrm{d}x_1, \mathrm{d}x_2) = (\mu_1 \wedge \mu_2)(\mathrm{d}x_1) I_\Delta + \frac{(\mu_1 - \mu_2)^+(\mathrm{d}x_1)(\mu_2 - \mu_1)^+(\mathrm{d}x_2)}{(\mu_1 - \mu_2)^+(E)} I_{\Delta^c},$$

$(\nu_1 - \nu_2)^\pm$: Jordan-Hahn decomposition. Ignore I_{Δ^c}. $\nu_1 \wedge \nu_2 = \nu_1 - (\nu_1 - \nu_2)^+$. $\Delta =$ diagonals.

$$\tilde{\mu}_b(\rho) = \frac{1}{2}\|\mu_1 - \mu_2\|_{\mathrm{Var}} = \inf_{\tilde{\mu}} \tilde{\mu}(\rho).$$

ρ: discrete distance. $\tilde{\mu}_b$:ρ-optimal coupling.
Markov processes: $P_k(t, x_k, \mathrm{d}y_k)$.
$P_k(t)$ on $(E_k, \mathscr{E}_k)$, $k = 1, 2$. $\widetilde{P}^{t;x_1,x_2}(A_1 \times A_2)$.

Markovian couplings

A Markov process $\widetilde{P}(t)$ on $(E_1 \times E_2, \mathscr{E}_1 \times \mathscr{E}_2)$ having marginality:

$$\begin{aligned}
&\widetilde{P}(t; x_1, x_2; A_1 \times E_2) = P_1(t, x_1, A_1),\\
&\widetilde{P}(t; x_1, x_2; E_1 \times A_2) = P_2(t, x_2, A_2), \qquad \text{(MP)}\\
&\quad t \geqslant 0,\ x_k \in E_k,\ A_k \in \mathscr{E}_k,\ k = 1,\ 2.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\widetilde{P}(t) f_1(x_1, x_2) = P_1(t) f(x_1),\\
&\widetilde{P}(t) f_2(x_1, x_2) = P_2(t) f(x_2), \qquad \text{(MP)}\\
&\quad t \geqslant 0,\ x_k \in E_k,\ f \in {}_b\mathscr{E}_k,\ k = 1,\ 2,
\end{aligned}$$

${}_b\mathscr{E}$: bounded $\mathscr{E}$-measurable functions.
On LHS, $f_1(x_1, x_2) = f(x_1)$, $f_2(x_1, x_2) = f(x_2)$.

Jump processes

Jump condition:

$\lim_{t\to 0} P(t,x,\{x\}) = 1, \qquad x \in E.$

q-pair $(q(x), q(x,\mathrm{d}y))$:

$$0 \leqslant q(x) = \lim_{t\to 0} \frac{1-P(t,x,\{x\})}{t} \leqslant \infty, \quad x \in E$$

$$q(x,A) = \lim_{t\to 0} \frac{P(t,x,A\setminus\{x\})}{t} \leqslant q(x), \quad x \in E,\ A \in \mathscr{R}$$

$$\mathscr{R} = \left\{ A \in \mathscr{E} :\ \lim_{t\to 0} \sup_{x\in A} \left[1 - P(t,x,\{x\})\right] = 0 \right\}.$$

Totally stable: $q(x) < \infty$ for all x.

Conservative: $q(x,E) = q(x)$ for all x.

Operator: $\Omega f(x) = \int q(x,\mathrm{d}y)[f(y) - f(x)],\ f \in {}_b\mathscr{E}.$

Coupling q-pair:

$$\tilde{q}(x_1, x_2) = \lim_{t \to 0} [1 - \widetilde{P}(t; x_1, x_2; \{x_1\} \times \{x_2\})]/t,$$
$$(x_1, x_2) \in E_1 \times E_2$$
$$\tilde{q}(x_1, x_2; \tilde{A}) = \lim_{t \to 0} [1 - \widetilde{P}(t; x_1, x_2; \tilde{A})]/t,$$
$$(x_1, x_2) \notin \tilde{A} \in \widetilde{\mathcal{R}}.$$

Theorem

- [C., 1994]. Coupling Markov process is a jump process $\iff$ so are the marginals.
- [C., 1994]. Coupling q-pair totally stable $\iff$ so are the marginals.
- [Y. H. Zhang, 1994]. Coupling q-pair conservative $\iff$ so are the marginals.

Marginality for operators:

Given Ω_1 and Ω_2, from (MP), it follows that any

$$\widetilde{\Omega} f(x_1, x_2) = \int_{E\times E} \tilde{q}(x_1, x_2; \mathrm{d}y_1, \mathrm{d}y_2)[f(y_1, y_2) - f(x_1, x_2)].$$

must satisfies

$$\begin{aligned} \widetilde{\Omega} f_1(x_1, x_2) &= \Omega_1 f(x_1), \qquad f \in {}_b\mathscr{E}_1 \\ \widetilde{\Omega} f_2(x_1, x_2) &= \Omega_2 f(x_2), \qquad f \in {}_b\mathscr{E}_2, \\ x_k \in E_k,\ k &= 1,\ 2. \end{aligned} \qquad \text{(MO)}$$

—coupling operator.

Markov chain: $P(t) = (P_{ij}(t) : i, j \in E)$.

Q-matrix: $Q = (q_{ij}) := \dfrac{\mathrm{d}}{\mathrm{d}t} P(t)\Big|_{t=0}$.

$Qf(i) := \sum_{j\in E} q_{ij} f_j = \sum_{j\neq i} q_{ij} [f_j - f_i]$.

$\widetilde{P}(t) \longrightarrow \widetilde{Q}$. Marginality for operators:

$$\begin{aligned} \widetilde{Q} f_1(i_1, i_2) &= Q_1 f(i_1) \\ \widetilde{Q} f_2(i_1, i_2) &= Q_2 f(i_2) \quad i_1, i_2 \in E. \end{aligned} \qquad \text{(MO)}$$

$\widetilde{Q}$: coupling Q-matrix of Q_1 and Q_2.

Question: Does there exists a coupling operator?

Examples of coupling for jump processes

Example 1. Independent coupling $\widetilde{\Omega}_0$:

$$\widetilde{\Omega}_0 f(x_1, x_2) = [\Omega_1 f(\cdot, x_2)](x_1) + [\Omega_2 f(x_1, \cdot)](x_2)$$
$$x_k \in E_k, \ k = 1, 2.$$

Example 2. Classical coupling [Doeblin, 1938] $\widetilde{\Omega}_c$:
$\Omega_1 = \Omega_2 = \Omega$.

$$\begin{aligned} x_1 \neq x_2, \quad (x_1, x_2) &\to (y_1, x_2) \ \text{ at rate } q(x_1, \mathrm{d}y_1) \\ &\to (x_1, y_2) \ \text{ at rate } q(x_2, \mathrm{d}y_2). \\ \text{Otherwise, } (x, x) &\to (y, y) \ \text{ at rate } q(x, \mathrm{d}y). \end{aligned}$$

Character. Chinese idiom: fall in love at first sight.
Omit the last property from now on.

Example 3. Basic coupling [Wasserstein, 1969] $\widetilde{\Omega}_b$:

For $x_1,\ x_2 \in E$,

$$\begin{aligned}(x_1,\ x_2) \mapsto (y,\ y) \quad & \text{at rate}\left[q_1(x_1,\cdot)\wedge q_2(x_2,\cdot)\right](\mathrm{d}y)\\ \to (y_1,\ x_2)\ & \text{at rate}\left[q_1(x_1,\cdot) - q_2(x_2,\cdot)\right]^+(\mathrm{d}y_1)\\ \to (x_1,\ y_2)\ & \text{at rate}\left[q_2(x_2,\cdot) - q_1(x_1,\cdot)\right]^+(\mathrm{d}y_2)\end{aligned}$$

$(\nu_1 - \nu_2)^{\pm}$: Jordan-Hahn decomposition.

$\nu_1 \wedge \nu_2 = \nu_1 - (\nu_1 - \nu_2)^+$.

Example 4. Coupling of marching soldiers [C., 1986] $\widetilde{\Omega}_m$. E: Addition group.

$$(x_1,\ x_2) \ \to\ (x_1 + y, x_2 + y)$$
$$\text{at rate}\ \left[q_1(x_1, x_1 + y) \wedge q_2(x_2, x_2 + y)\right](\mathrm{d}y).$$

Marching: A Chinese command to soldiers to start marching.

Birth-death Q-matrix:

$$\begin{aligned} i \;\to\; & i+1 \quad \text{at rate} \quad b_i = q_{i,i+1} \\ \to\; & i-1 \quad \text{at rate} \quad a_i = q_{i,i-1}. \end{aligned}$$

Example 5. Coupling by inner reflection [C., 1990]

$\widetilde{\Omega}_{ir}$:

Take $\widetilde{\Omega}_{ir} = \widetilde{\Omega}_c$ if $|i_1 - i_2| \leqslant 1$. For $i_2 \geqslant i_1 + 2$, take

$$\begin{aligned} (i_1,\, i_2) \;\to\; & (i_1+1,\, i_2-1) \quad \text{at rate} \quad b_{i_1} \wedge a_{i_2} \\ \to\; & (i_1-1,\, i_2) \quad \text{at rate} \quad a_{i_1} \\ \to\; & (i_1,\, i_2+1) \quad \text{at rate} \quad b_{i_2}. \end{aligned}$$

Exchange i_1 and i_2 to get $\widetilde{\Omega}_{ir}$ for $i_1 \geqslant i_2 + 2$.

Infinite many choices of coupling operator $\widetilde{\Omega}$!
Non-explosive?

Fundamental Theorem for Couplings of Jump Processes [C., 1986].

- If a coupling operator is non-explosive, then so are their marginals.
- If the marginals are both non-explosive, then so is every coupling operator.
- If so, then (MP)$\Longleftrightarrow$(MO).

Markovian couplings for diffusions

Couplings for elliptic operators in $\mathbb{R}^d$.

$$L = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}.$$

An elliptic (may be degenerated) operator $\widetilde{L}$ on the product space $\mathbb{R}^d \times \mathbb{R}^d$ is called a coupling of L if it satisfies the following marginality:

$$\widetilde{L} f_1(x,y) = Lf(x) \text{ (resp. } \widetilde{L} f_2(x,y) = Lf(y)),$$
$$f \in C_b^2(\mathbb{R}^d), \quad x \neq y.$$

(MO)

Coefficients of $\widetilde{L}$:

$$a(x,y)=\begin{pmatrix} a(x) & c(x,y) \\ c(x,y)^* & a(y) \end{pmatrix}, \quad b(x,y)=\begin{pmatrix} b(x) \\ b(y) \end{pmatrix}.$$

Condition: $a(x,y)$ non-negative definite.
Only freedom: $c(x,y)$.

Three examples:
(1) Classical coupling $c(x,y)\equiv 0, x\neq y$.
(2) Coupling of marching soldiers [C. and S. F. Li,1989].
Let $a(x)=\sigma(x)^2$. Take $c(x,y)=\sigma(x)\sigma(y)$.

The couplings given below are due to Lindvall & Rogers (1986) and C. & Li (1989) respectively.

(3) Coupling by reflection. Take

$$c(x,y)=\sigma(x)\left[\sigma(y)^*-2\frac{\sigma(y)^{-1}\bar{u}\bar{u}^*}{|\sigma(y)^{-1}\bar{u}|^2}\right],$$

$$\det\sigma(y)\neq 0,\qquad x\neq y$$

$$c(x,y)=\sigma(x)\left[I-2\bar{u}\bar{u}^*\right]\sigma(y)^*,\ x\neq y$$

where $\bar{u}=(x-y)/|x-y|$. Extended to manifold by W.S.Kendall[1986]. Also M. Cranston[1991].

In the case that $x=y$, the first and the third ones are defined to be the same as the second one.

Reduce higher dim. to dim. one.

Couplings by stochastic differential equations.

$$\mathrm{d}X_t = \sqrt{2}\sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t$$
$$\mathrm{d}X'_t = \sqrt{2}\sigma'(X_t)\mathrm{d}B'_t + b'(X_t)\mathrm{d}t$$

- Classical coupling: B_t and B'_t are independent.
- Coupling of marching soldiers: $B_t = B'_t$
- Coupling by reflection: $B'_t = [I - 2\bar{u}\bar{u}^*](X_t, X'_t)B_t$, where $\bar{u}$ is given in the last page.

Conjecture: The fundamental theorem holds for diffusions. Facts:

- Sufficient condition for well-posed. There exists φ_k such that $\lim_{|x|\to\infty}\varphi_k(x) = \infty$ and $L\varphi \leqslant c\varphi$ for some constant c. Then, take $\tilde\varphi(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2)$.
- Let $\tau_{n,k}$ be the first time leaving from the cube with length n of the k-th process and let $\tilde\tau_n$ be the first time leaving the product cube of coupled process, then we have $\tau_{n,1} \vee \tau_{n,2} \leqslant \tilde\tau_n \leqslant \tau_{n,1} + \tau_{n,2}$. Well-posed iff $\lim_{n\to\infty}\mathbb{P}_k[\tau_{n,k} < t] = 0$.

Problem: Markovian couplings for Levý processes.

2. Optimal Markovian coupling w.r.t. distances, ρ-optimal Markovian couplings

There ∞ many Markovian couplings. Does there exist an optimal one?

For birth-death processes, we have an order as follows:

$$\widetilde{\Omega}_{ir} \succ \widetilde{\Omega}_b \succ \widetilde{\Omega}_c \succ \widetilde{\Omega}_m,$$

Probability distances: $\geqslant 16$, the total variation, Lévy-Prohorov distance for the weak convergence. Another probability distance.

Typical types of convergence in probability theory

$$
\begin{array}{ccccc}
 & & \text{convergence in } L^p & & \\
 & & \Downarrow & & \\
\text{a.s. conv.} & \Rightarrow & \text{convergence in } \mathbb{P} & \Rightarrow & \text{vague conv.} \\
 & \Nwarrow & & \Nearrow\!\!\!\Swarrow & \\
 & & \text{weak convergence} & &
\end{array}
$$

L^p-convergence, a.s. convergence and convergence in $\mathbb{P}$ depend on the reference frame: $(\Omega, \mathscr{F}, \mathbb{P})$. Vague (weak) convergence does not.

Skorohod Thm. (cf. Ikeda and Watanabe (1981), p.9 Thm.2.7): if $P_n \Longrightarrow P$, choose reference frame $(\Omega, \mathscr{F}, \mathbb{P})$ such that $\xi_n \sim P_n$, $\xi \sim P$, $\xi_n \to \xi$ a.s. Except the L^p-convergence.

Let ξ_1, ξ_2: $(\Omega, \mathscr{F}, \mathbb{P}) \to (E, \rho, \mathscr{E})$. L^p-distance:

$$\|\xi_1 - \xi_2\|_p = \left\{\mathbb{E}\left[\rho(\xi_1, \xi_2)^p\right]\right\}^{1/p}, \quad p \geqslant 1.$$

Let $\xi_i \sim P_i$, $i = 1, 2$ and $(\xi_1, \xi_2) \sim \widetilde{P}$. Then

$$\|\xi_1 - \xi_2\|_p = \left\{ \int \rho(x_1, x_2)^p \widetilde{P}(dx_1, dx_2) \right\}^{1/p}.$$

$\widetilde{P}$: a coupling of P_1 and P_2. Ignore our reference frame $(\Omega, \mathscr{F}, \mathbb{P})$, then there are a lot of choices of $\widetilde{P}$ for given P_1 and P_2. Thus, the intrinsic distance should be defined as follows:

$$W_p(P_1, P_2) = \inf_{\widetilde{P}} \left\{ \int \rho(x_1, x_2)^p \widetilde{P}(dx_1, dx_2) \right\}^{1/p}$$

—p-th Wasserstein distance.

Intrinsic: $\xi = \eta + x$, $W_p(P_\xi, P_\eta) = |x|$.
Set $W = W_1$. W in dim. one. W_2 for Gaussian.
Dobrushin Theorem: For discrete ρ, $W = \|\cdot\|_{\text{var}}/2$.
Definition: Coupling $\overline{P}$ of P_1 and P_2 is called ρ-optimal if

$$\int \rho(x_1, x_2)\overline{P}(\mathrm{d}x_1, \mathrm{d}x_2) = W(P_1, P_2).$$

Time-discrete proc. and discrete ρ, Griffeath (1975): maximal coupling. Non-Markovian!
Coupling from the past:
(J. G. Propp & D. B. Wilson, 1996).

Definition: $(E, \rho, \mathscr{E})$ metric space. Coupling operator $\overline{\Omega}$ is called ρ-optimal if
$\overline{\Omega}\,\rho(x_1, x_2) = \inf_{\widetilde{\Omega}} \widetilde{\Omega}\rho(x_1, x_2)$ for all $x_1 \neq x_2$.

Example 6 (Coupling by reflection) [C., 1994].

If $i_2 = i_1 + 1$, then

$$\begin{aligned} (i_1, i_2) &\to (i_1 - 1, i_2 + 1) &&\text{at rate } a_{i_1} \wedge b_{i_2} \\ &\to (i_1 + 1, i_2) &&\text{at rate } b_{i_1} \\ &\to (i_1, i_2 - 1) &&\text{at rate } a_{i_2}. \end{aligned}$$

If $i_2 \geqslant i_1 + 2$, then

$$\begin{aligned} (i_1, i_2) &\to (i_1 - 1, i_2 + 1) &&\text{at rate } a_{i_1} \wedge b_{i_2} \\ &\to (i_1 + 1, i_2 - 1) &&\text{at rate } b_{i_1} \wedge a_{i_2}. \end{aligned}$$

$i_1 > i_2$: by symmetry. Reflect. outside strange.

There exist infinitely many choices of $\widetilde{Q}$!

Theorem [C., 1994]. Birth-death processes. Given positive (u_i).

- Take distance $\rho(i,j) = \sum_{k<|i-j|} u_k$. If u_k is non-increasing in k, then the coupling by reflection is ρ-optimal.
- Take ρ as above. If u_k is non-decreasing in k, then the coupling of marching soldiers is ρ-optimal.
- Take $\rho(i,j) = \left|\sum_{k<i} u_k - \sum_{k<j} u_k\right|$. Then the above couplings, except the independent one, are all ρ-optimal.

Far away from probabilistic intuition.

Definition. Given $\rho \in C^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\})$, a coupling operator $\overline{L}$ is called *ρ-optimal* if

$$\overline{L}\rho(x,y) = \inf_{\widetilde{L}} \widetilde{L}\rho(x,y), \quad x \neq y,$$

where $\widetilde{L}$ varies over all coupling operator.

Theorem [C., 1994]. Let $f \in C^2(\mathbb{R}_+; \mathbb{R}_+)$: $f(0) = 0$, $f' > 0$, $f'' \leqslant 0$. Set $\rho(x,y) = f(|x-y|)$. Then, the ρ-optimal solution $c(x,y)$ is given as follows.

- If $d = 1$, then $c(x,y) = -\sqrt{a_1(x)a_2(y)}$ and moreover,

$$\overline{L}f(|x-y|) = \frac{1}{2}\big(\sqrt{a_1(x)} + \sqrt{a_2(y)}\,\big)^2 f''(|x-y|) + \frac{(x-y)(b_1(x)-b_2(y))}{|x-y|} f'(|x-y|).$$

Next, suppose that $a_k = \sigma_k^2$ $(k = 1,\ 2)$ is non-degenarated and write

$$c(x,y) = \sigma_1(x) H^*(x,y) \sigma_2(y).$$

- If $f''(r) < 0$ for all $r > 0$, then $H(x,y) = U(\gamma)^{-1}\big[U(\gamma)U(\gamma)^*\big]^{1/2}$, where

$$\gamma = 1 - \frac{|x-y| f''(|x-y|)}{f'(|x-y|)}, U(\gamma) = \sigma_1(x)(I - \gamma\bar{u}\bar{u}^*)\sigma_2(y).$$

- If $f(r) = r$, then $H(x, y)$ is a solution to the equation:

$$U(1)H = \big(U(1)U(1)^*\big)^{1/2}.$$

In particular, if $a_k(x) = \varphi_k(x)\sigma^2$ for some positive function φ_k ($k = 1$, 2), where σ is independent of x and $\det\sigma > 0$. Then

- when $\rho(x, y) = |x - y|$,
  $H(x, y) = I - 2\sigma^{-1}\bar{u}\bar{u}^*\sigma^{-1}/|\sigma^{-1}\bar{u}|^2$.

- In the last term, one can replace $\rho(x, y) = |x - y|$ by $\rho(x, y) = f(|\sigma^{-1}(x - y)|)$.

3. Optimal Markovian coupling w.r.t. nonnegative, lower semi-continuous functions. φ-optimal Markovian couplings

Given metric space $(E, \rho, \mathscr{E})$. φ: a nonnegative, lower semi-continuous function.

Definition: φ-optimal (Markovian) coupling, replace ρ with φ.

Examples:

- $\varphi = f \circ \rho$, f: $f(0) = 0$, $f' > 0$, $f'' \leqslant 0$. Cont.
- $\varphi(x, y) = 1$ iff $x \neq y$, otherwise, $\varphi(x, y) = 0$.
- Let E have a measurable semi-order "$\leqslant$" and $F := \{(x, y) : x \leqslant y\}$ is a closed set. Take $\varphi = I_{F^c}$.

Definition. Let $\mathscr{M}$ be set of bdd monotone functions f: $x \leqslant y \Longrightarrow f(x) \leqslant f(y)$.

- $\mu_1 \leqslant \mu_2$: $\mu_1(f) \leqslant \mu_2(f), \forall f \in \mathscr{M}$.
- $P_1 \leqslant P_2$: $P_1(f)(x_1) \leqslant P_2(f)(x_2)$, $\forall x_1 \leqslant x_2, f \in \mathscr{M}$.
- $P_1(t) \leqslant P_2(t)$: $P_1(t)(f)(x_1) \leqslant P_2(t)(f)(x_2)$, $\forall t \geqslant 0, x_1 \leqslant x_2, f \in \mathscr{M}$.

Theorem [V. Strassen, 1965]. For Polish space, $\mu_1 \leqslant \mu_2 \iff \exists$ a coupling measure $\bar{\mu}$ such that $\bar{\mu}(F^c) = 0$.

Existence theorem of optimal couplings [S. Y. Zhang, 2000].

Let $(E, \rho, \mathscr{E})$ be Polish and $\varphi \geqslant 0$ be l.s.c.

- Given $P_k(x_k, \mathrm{d}y_k)$, $k = 1, 2$, there exists $\overline{P}(x_1, x_2; \mathrm{d}y_1, \mathrm{d}y_2)$ such that $\overline{P}\varphi = \inf_{\widetilde{P}} \widetilde{P}\varphi$.
- Given jump operators Ω_k, $k = 1, 2$, there exists a coupling operator $\overline{\Omega}$ such that $\overline{\Omega}\varphi = \inf_{\widetilde{\Omega}} \widetilde{\Omega}\varphi$.

Strassen's theorem: I_{F^c}-optimal Markovian coupling satisfies $\bar{\mu}(F^c) = 0$.

Theorem [C., 1992, Y. H. Zhang, 2000]
Polish space. Jump processes.
$P_1(t) \leqslant P_2(t)$ iff

$$\Omega_1 I_B(x_1) \leqslant \Omega_2 I_B(x_2), \quad \forall x_1 \leqslant x_2, B \in \mathscr{M}.$$

Theorem [T. Lindvall, 1999]. Δ: diagonals.

- $\mu_1 \leqslant \mu_2 \Longrightarrow \inf_{\tilde\mu(F^c)=0} \tilde\mu(\Delta^c) = \frac{1}{2}\|\mu_1 - \mu_2\|_{\text{Var}}$.
- Let $P_1 \leqslant P_2$. Then

$$\inf_{\widetilde{P}(x_1,x_2;F^c)=0} \widetilde{P}(x_1, x_2; \Delta^c) = \frac{1}{2}\|P_1(x_1,\cdot) - P_2(x_2,\cdot)\|_{\text{Var}}$$

for all $x_1 \leqslant x_2$.

For diffusions, "$P_1(t) \leqslant P_2(t)$", solved.

Problem: Existence of φ-optimal Markovian couplings for diffusions. $\varphi \in C^2(\mathbb{R}^{2d} \setminus \Delta)$.
Problem: Construction of optimal Markovian couplings.

New Variational Formulas for the First Eigenvalue

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(November 15, 2002)

Contents

- Introduction.
- Variational formulas of the first eigenvalue.
- Further applications.

I. Introduction

1.1 Definition. The first (non-trivial) eigenvalue:

$$Q = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$a_i > 0, \; b_i > 0.$$

$Q1 = 0$. Trivial eigenvalue: $\lambda_0 = 0$.

Question: Next eigenvalue of $-Q$: $\lambda_1 = ?$

Elliptic operator in $\mathbb{R}^d$; Laplacian on Riemannian manifolds. Importance: leading term.

1.2 Motivation (Applications)

(1) Phase transitions

$$L = L_\beta,\ P_t = e^{tL}$$

$$\|P_t - \pi\|_{2\to 2} \le e^{-\lambda_1 t}$$

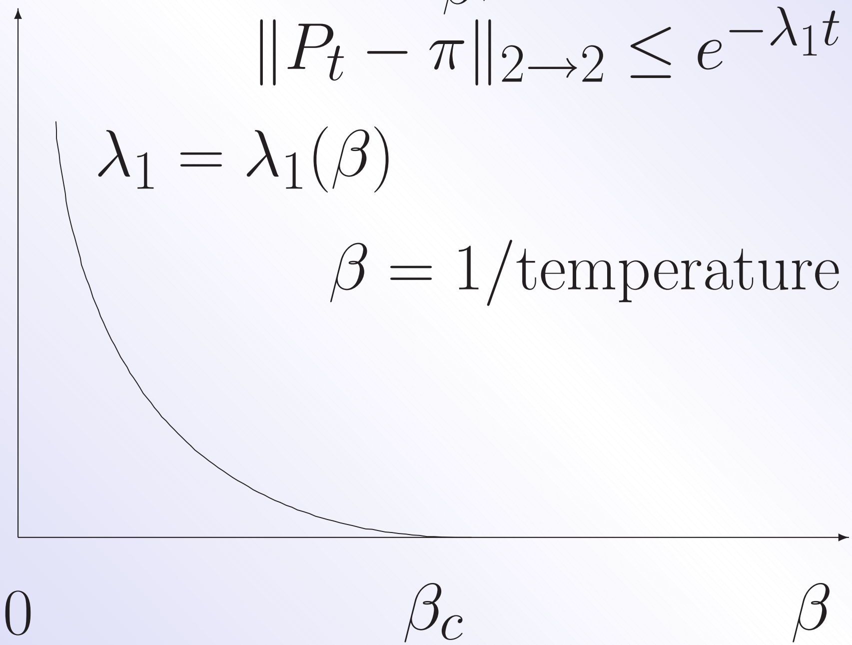

Figure 1: Phase transition and λ_1

(2) Random algorithm: Markov Chains Monte Carlo

NP$\Longrightarrow$P. S. Smale (80's). A. Sinclair (93)

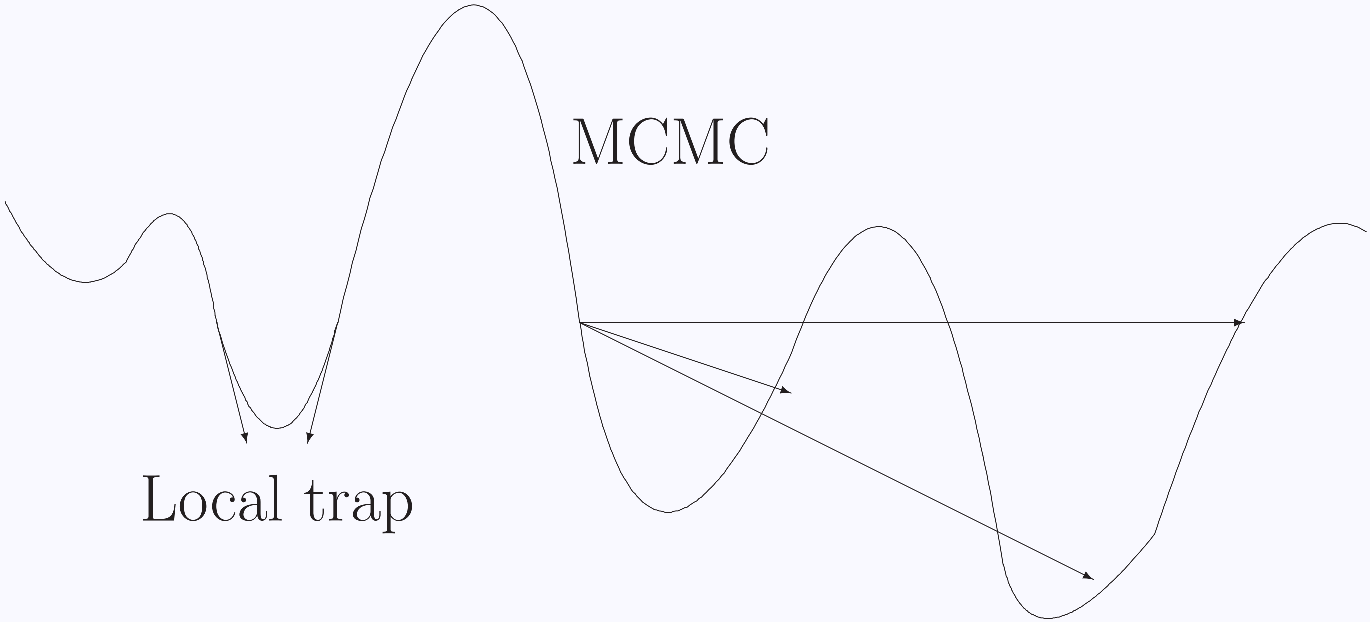

Figure 2: Random algorithm and λ_1

Travelling Salesman Problem:

Find shortest closed path (without loop) among 144 cities in China.

For computer with speed 10^9 path/second:

$$\frac{143!}{10^9 \times 365 \times 24 \times 60 \times 60} \approx 10^{231} \text{ (years)}.$$

Typical NP-problem!

MCMC: 30421 km. Best known: 30380 km.

- Markov chain.
- Stay at lower place with bigger probability. Gibbs principle.
- Algorithm with bigger λ_1 is more effective!

1.3 Difficulties

Example 1: Trivial case(two points). Two parameters.

$$\begin{pmatrix} -b & b \\ a & -a \end{pmatrix}, \quad \lambda_1 = a + b.$$

λ_1 is increasing in each of the parameters!

Example 2: Three points. Four parameters.

$$\begin{pmatrix} -b_0 & b_0 & 0 \\ a_1 & -(a_1 + b_1) & b_1 \\ 0 & a_2 & -a_2 \end{pmatrix},$$

$$\lambda_1 = 2^{-1}\Big[a_1 + a_2 + b_0 + b_1 \\ -\sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1b_1}\Big].$$

Example 3: Four points.

Six parameters: $b_0, b_1, b_2, a_1, a_2, a_3$.

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3}\,\left(3\,B - D^2\right)}{3\,C},$$

where

$$D = a_1 + a_2 + a_3 + b_0 + b_1 + b_2,$$

$$B = a_3\,b_0 + a_2\,\left(a_3 + b_0\right) + a_3\,b_1 + b_0\,b_1 + b_0\,b_2 + b_1\,b_2 + a_1\,\left(a_2 + a_3 + b_2\right),$$

$$C = \left(A + \sqrt{4\left(3\,B - D^2\right)^3 + A^2}\right)^{1/3},$$

$$A = -2\,a_1^3 - 2\,a_2^3 - 2\,a_3^3 + 3\,a_3^2\,b_0 + 3\,a_3\,b_0^2 - 2\,b_0^3 + 3\,a_3^2\,b_1 - 12\,a_3\,b_0\,b_1 + 3\,b_0^2\,b_1 + 3\,a_3\,b_1^2 + 3\,b_0\,b_1^2 - 2\,b_1^3 - 6\,a_3^2\,b_2 + 6\,a_3\,b_0\,b_2 + 3\,b_0^2\,b_2 + 6\,a_3\,b_1\,b_2 - 12\,b_0\,b_1\,b_2 + 3\,b_1^2\,b_2 - 6\,a_3\,b_2^2 + 3\,b_0\,b_2^2 + 3\,b_1\,b_2^2 - 2\,b_2^3 + 3\,a_1^2\,(a_2 + a_3 - 2\,b_0 - 2\,b_1 + b_2) + 3\,a_2^2\,[a_3 + b_0 - 2\,(b_1 + b_2)] + 3\,a_2\left[a_3^2 + b_0^2 - 2\,b_1^2 - b_1\,b_2 - 2\,b_2^2 - a_3(4\,b_0 - 2\,b_1 + b_2) + 2\,b_0(b_1 + b_2)\right] + 3\,a_1\left[a_2^2 + a_3^2 - 2\,b_0^2 - b_0\,b_1 - 2\,b_1^2 - a_2(4\,a_3 - 2\,b_0 + b_1 - 2\,b_2) + 2\,b_0\,b_2 + 2\,b_1\,b_2 + b_2^2 + 2\,a_3(b_0 + b_1 + b_2)\right].$$

The role of each parameter is completely mazed!

Not solvable when space has more than five points!

Conclusion: Impossible to compute λ_1 explicitly!

Perturbation of eigenvalues

Example: Infinite tri-diagonal matrix (Birth-death processes).

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i + \beta$ $(\beta > 0)$	$2i$	1	1
$i + 1$	$2i + 3$		
$i + 1$	$2i + \left(4 + \sqrt{2}\right)$		

Perturbation of eigenvalues

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$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	
$i+1$	$2i+\left(4+\sqrt{2}\right)$	3	

Perturbation of eigenvalues

Example: Infinite tri-diagonal matrix
(Birth-death processes).

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	2
$i+1$	$2i+\left(4+\sqrt{2}\right)$	3	3

Sensitive. In general, it is too hard to estimate λ_1!

II. New variational formulas of the first eigenvalue

2.1 Story of the study on λ_1 in geometry

(M, g): compact Riemannian manifold.

Discrete spectrum: $0 = \lambda_0 < \lambda_1 < \cdots.$

- g: Riemannian metric.
- d: dimension. $\mathbb{S}^d$: $D=\pi$, $\text{Ric}=d-1$.
- D: diameter.
- $\text{Ricci}_M \geqslant K g$ for some $K \in \mathbb{R}$.

Idea: Use geometric quantities d, D and K to estimate λ_k's of Laplacian Δ.

Five books!

- Chavel, I. (1984): Eigenvalues in Riemannian Geometry, Academic Press
- Bérard, P. H. (1986): Spectral Geometry: Direct and Inverse Problem, LNM. vol. 1207, Springer-Verlag. Including 2000 references.
- Schoen, R. and Yau, S. T. (1988): Differential Geometry (In Chinese), Science Press, Beijing, China
- Li, P. (1993): Lecture Notes on Geometric Analysis, Seoul National U., Korea
- Ma, C. Y. (1993): The Spectrum of Riemannian Manifolds (In Chinese), Press of Nanjing U., Nanjing

Ten of the most beautiful lower bounds:

Author(s)	Lower bound: $K \geqslant 0$
A. Lichnerowicz (1958)	$\frac{d}{d-1} K$
P. H. Bérard, G. Besson & S. Gallot (1985)	$d \left\{ \frac{\int_0^{\pi/2} \cos^{d-1} t \mathrm{d}t}{\int_0^{D/2} \cos^{d-1} t \mathrm{d}t} \right\}^{2/d}$ $K = d - 1$
P. Li & S. T. Yau (1980)	$\frac{\pi^2}{2 D^2}$
J. Q. Zhong & H. C. Yang (1984)	$\frac{\pi^2}{D^2}$
D. G. Yang (1999)	$\frac{\pi^2}{D^2} + \frac{K}{4}$

Author(s)	Lower bound: $K \leqslant 0$
P. Li & S. T. Yau (1980)	$\dfrac{1}{D^2(d-1)\exp\left[1+\sqrt{1+16\alpha^2}\right]}$
K. R. Cai (1991)	$\dfrac{\pi^2}{D^2}+K$
D. Zhao (1999)	$\dfrac{\pi^2}{D^2}+0.52K$
H. C. Yang (1989) & F. Jia (1991)	$\dfrac{\pi^2}{D^2}e^{-\alpha}, \quad \text{if } d \geqslant 5$
H. C. Yang (1989) & F. Jia (1991)	$\dfrac{\pi^2}{2\,D^2}e^{-\alpha'}, \quad \text{if } 2 \leqslant d \leqslant 4$

$$\alpha = D\sqrt{|K|(d-1)}/2,$$
$$\alpha' = D\sqrt{|K|((d-1)\vee 2)}/2.$$

2.2 New variational formulas

Theorem [C. & F. Y. Wang, 1997].

$$\lambda_1 \geqslant \sup_{f\in\mathscr{F}} \inf_{r\in(0,D)} \frac{4f(r)}{\int_0^r C(s)^{-1}\mathrm{d}s \int_s^D C(u)f(u)\mathrm{d}u}$$

Two notations:

$$C(r) = \cosh^{d-1}\left[\frac{r}{2}\sqrt{\frac{-K}{d-1}}\right], \quad r \in (0, D).$$

$$\mathscr{F} = \{f \in C[0, D] : f > 0 \text{ on } (0, D)\}.$$

Classical variational formula:

$$\lambda_1 = \inf\{\textstyle\int_M \|\nabla f\|^2 : f\in C^1, \pi(f)=0, \pi(f^2)=1\}$$

Goes back to Lord S. J. W. Rayleigh(1877) or E. Fischer (1905). Gen. R. Courant (1924).

Elementary functions: $1, \sin(\alpha r), \cosh^{1-d}(\alpha r)\sin(\beta r)$,

$$\alpha = D\sqrt{|K|/(d-1)}/2, \quad \beta = \frac{\pi}{2D}.$$

Corollary 1 [C. & F. Y. Wang, 1997].

$$\lambda_1 \geqslant \frac{dK}{d-1}\left\{1-\cos^d\left[\frac{D}{2}\sqrt{\frac{K}{d-1}}\right]\right\}^{-1}, \quad d>1, K\geqslant 0.$$

$$\lambda_1 \geqslant \frac{\pi^2}{D^2}\sqrt{1-\frac{2D^2K}{\pi^4}\cosh^{1-d}\left[\frac{D}{2}\sqrt{\frac{-K}{d-1}}\right]},$$

$$d>1, \quad K\leqslant 0.$$

Corollary 2 [C., E. Scacciatelli and L. Yao, 2002].

$$\lambda_1 \geqslant \frac{\pi^2}{D^2}+\frac{K}{2}, \qquad K\in\mathbb{R}.$$

Representative test function:

$$f(r) = \left(\int_0^r C(s)^{-1}\mathrm{d}s\right)^\gamma, \quad \gamma = \frac{1}{2}, 1.$$

$$C(s) = \cosh^{d-1}\left[\frac{s}{2}\sqrt{\frac{-K}{d-1}}\right].$$

$$\delta = \sup_{r\in(0,D)} \left(\int_0^r C(s)^{-1}\mathrm{d}s\right)\left(\int_r^D C(s)\mathrm{d}s\right).$$

Corollary 3 [C., 2000]. $\lambda_1 \geqslant \xi_1$.

$$4\delta^{-1} \geqslant (\delta_n')^{-1} \geqslant \xi_1 \geqslant \delta_n^{-1} \geqslant \delta^{-1},$$

$$\text{Explicit } (\delta_n')^{-1}\downarrow, \quad \delta_n^{-1}\uparrow.$$

Convex ∂M. Lichnerowicz. J. F. Escobar (1990).

2.3 "Proof": Estimation of λ_1

Step 1. g: eigenfunction: $Lg = -\lambda_1 g$, $g \neq$const.

From semigroups theory,

$$\frac{d}{dt}T_t g(x) = T_t Lg(x) = -\lambda_1 T_t g(x).$$

ODE. $T_t g(x) = g(x)e^{-\lambda_1 t}$. (1)

Step 2. Compact space. g Lipschitz w.r.t. ρ: c_g.

Key condition: $\widetilde{T}_t \rho(x,y) \leqslant \rho(x,y)e^{-\alpha t}$. (2)

$$\begin{aligned} e^{-\lambda_1 t}|g(x)-g(y)| &\leqslant \widetilde{T}_t\big|g(x)-g(y)\big| && \text{(by (1))}\\ &\leqslant c_g \widetilde{T}_t\rho(x,y)\\ &\leqslant c_g\rho(x,y)e^{-\alpha t} && \text{(by (2))}\end{aligned}$$

for all t. Hence $\lambda_1 \geqslant \alpha$. General!

$$(2) \iff \widetilde{L}\rho(x,y) \leqslant -\alpha\rho(x,y).$$

Two key points

• “Good” coupling: Classification of couplings. ρ-optimal couplings. “optimal mass transportation”.

• “Good” distance: Ord.$\Rightarrow$ none. Reduce to dim. one.

Redesignated distances

Diffusion on $[0,\infty)$. $L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$.
Closely related to eigenfunction! Let $b(x) \equiv 0$.

$$
\begin{aligned}
& a(x)g'' = -\lambda_1 g \\
\iff & g'(s) = \int_s^\infty \frac{\lambda_1 g}{a} \quad \text{(since } g'(0) = 0\text{)} \\
\iff & g(x) = g(0) + \int_0^x \mathrm{d}s \int_s^\infty \frac{\lambda_1 g}{a}.
\end{aligned}
$$

- Regard $\lambda_1 g$ as a new function f.
- Regard the right-hand side as an approximation of the left-hand side g.
- Ignore the constant $g(0)$ on the right-hand side since we are interested only in $g(x) - g(y)$.

Mimic of eigenfunction:

$$\tilde g(x) = \int_0^x \mathrm{d}s \int_s^\infty \frac{f}{a}, \quad \tilde g \uparrow\uparrow. \qquad f\text{: test function.}$$

Then define distance: $\rho(x, y) = |\tilde g(x) - \tilde g(y)|$.

For general $L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$,

By standard ODE,

$$\tilde g(x) = \int_0^x e^{-C(s)} \mathrm{d}s \int_s^\infty \frac{f e^C}{a}, \qquad C(r) := \int_0^r \frac{b}{a}$$

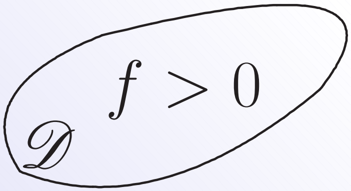
Complete variational formula in dimension one.

Higher dimension. Manifolds. Role of coupling!

Convergence for strong ergodicity.

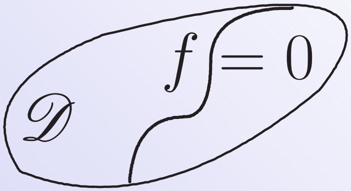
Intrinsic reason of success

Interaction of Prob Theory & Analysis, Geometry.



Dirichlet: $f(\partial\mathscr{D}) = 0$, 99%

$\Longleftrightarrow$ Maximum principle.



Neumann: $\partial f/\partial\mathbf{n}=0, \int f \mathrm{d}\pi=0.$

Surface $\{f=0\}$ depends on L, $\mathscr{D}$

Coupling method: $\rho(X_t, Y_t)$, degenerated at T.

Need only $\{t < T\}$. Do not care about position X_T.

Geometric Proof

$f \sim \lambda_1. \quad 1 = \sup f > \inf f =: -k.$

- Li-Yau's key estimate:

$$|\nabla f| \leqslant \frac{2\lambda_1}{1+k}(1-f)(k+f).$$

- Zhong-Yang's (Yang's, Cai's,...) key estimate:

$$|\nabla \theta|^2 \leqslant \lambda_1\big(1 + a_\varepsilon \psi(\theta)\big).$$

$$\theta = \arcsin(\text{a linear function of } f),$$

$$a_\varepsilon = \frac{1-k}{(1+k)(1+\varepsilon)},$$

$$\psi(\theta) = \begin{cases} \dfrac{\frac{2}{\pi}\left(2\theta + \sin(2\theta)\right) - 2\sin\theta}{\cos^2\theta}, & \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \\ 1, & \theta = \dfrac{\pi}{2} \\ -1, & \theta = -\dfrac{\pi}{2}. \end{cases}$$

Maximum Principle.

III. Further applications

3.1 Diffusion processes on $[0,\infty)$

Notations:

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad C(x) = \int_0^x \frac{b}{a},$$

$$\mu(\mathrm{d}x) = \frac{e^{C(x)}}{a(x)}\mathrm{d}x, \qquad \pi(\mathrm{d}x) = \mu(\mathrm{d}x)/\mu[0,\infty).$$

$$\mathscr{F} = \{f \in C[0,\infty) \cap C^1(0,\infty) : f(0) = 0, f'|_{(0,\infty)} > 0\}$$

$\mathscr{F}' =$ A slight modification of $\mathscr{F}$.

Theorem [C., 1996, 2000, 2001].

- Dual variational formulas. Write $\bar f = f - \pi(f)$.
  $$\lambda_1 \geqslant \sup_{f\in\mathscr{F}} \inf_{x>0} e^{C(x)} f'(x) \Big/ \int_x^\infty \bar f e^C / a.$$
  $$\lambda_1 \leqslant \inf_{f\in\mathscr{F}'} \sup_{x>0} e^{C(x)} f'(x) \Big/ \int_x^\infty \bar f e^C / a.$$
  Equalities hold for continuous a and b.
- Explicit estimates. $\delta(\theta)^{-1} \geqslant \lambda_1 \geqslant (4\delta(\theta))^{-1}$ for $\theta \in (0,\infty)$, where $\delta(\theta) = \sup_{x>\theta} \int_\theta^x e^{-C} \int_x^\infty e^C / a$.
- Approximation procedure. $\exists$ explicit η_n', η_n'' such that $\eta_n'^{-1} \geqslant \lambda_1 \geqslant \eta_n''^{-1} \geqslant (4\delta)^{-1}$.

3.2 Tri-diagonal matrix (birth-death process)

Notations:

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geqslant 1.$$

$$Z = \sum_i \mu_i < \infty, \qquad \pi_i = \frac{\mu_i}{Z}.$$

$\mathscr{F} = \{f : f_0 = 0, \ f$ is strictly increasing$\}$,

$\mathscr{F}' =$ A slight modification of $\mathscr{F}$.

Theorem [C., 1996, 2000, 2001].

- **Dual variational formulas.** Write $\bar f = f - \pi(f)$.

  $$\lambda_1 = \sup_{f\in\mathscr{F}} \inf_{i\geqslant 0} \mu_i b_i(f_{i+1} - f_i)\Big/ \sum\nolimits_{j\geqslant i+1} \mu_j \bar f_j.$$

  $$\lambda_1 = \inf_{f\in\mathscr{F}'} \sup_{i\geqslant 1} \mu_i b_i(f_{i+1} - f_i)\Big/ \sum\nolimits_{j\geqslant i+1} \mu_j \bar f_j.$$

- **Explicit estimates.** $Z\delta^{-1} \geqslant \lambda_1 \geqslant (4\delta)^{-1}$, where $\delta = \sup_{i\geqslant 1} \sum_{j\leqslant i-1} (\mu_j b_j)^{-1} \sum_{j\geqslant i} \mu_j$.

- **Approximation procedure.** $\exists$ explicit η_n', η_n'' such that $\quad {\eta_n'}^{-1} \geqslant \lambda_1 \geqslant {\eta_n''}^{-1} \geqslant (4\delta)^{-1}$.

3.3 Others

- The rate of strong ergodicity for diffusion on compact manifold.

  Theorem [Y. H. Mao, 2002]:

$$\text{Rate} \geqslant 4\bigg[\int_0^D C(s)^{-1}\text{d}s\int_s^D C(u)\text{d}u\bigg]^{-1}.$$

- $\cdots\cdots$ C. (1997): Three reports, 14 problems.

  Problem. Estimating λ_1 for complex manifolds by couplings.

Generalized Cheeger's Method

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(November 29, 2002)

Contents

- Cheeger's method.
- A generalization.
- New results.
- Splitting technique and existence criterion.
- Sketch of proofs of one main Theorem.

1. Cheeger's method

Probability $\longrightarrow$ Riemannian Geometry;
Riemannian Geometry $\longrightarrow$ Probability.
M: Compact Riemannian manifold. Laplacian Δ.
The first non-trivial eigenvalue λ_1. Dirichlet λ_0.
Theorem (Cheeger's inequality, 1970).

$$k \geqslant \lambda_1 \geqslant \frac{1}{4}k^2,$$

$$k := \inf_{M_1, M_2:\, M_1 \cup M_2 = M} \frac{\text{Area}(M_1 \cap M_2)}{\text{Vol}(M_1) \wedge \text{Vol}(M_2)}$$

— Cheeger's constant.

Key ideas:

- **Splitting technique**: $\lambda_1 \geqslant \inf_B[\lambda_0(B) \vee \lambda_0(B^c)]$.
- Estimate $\lambda_0(B)$: $h = \inf_{M_1 \subset M,\, \partial M_1 \cap \partial M = \emptyset} \frac{\text{Area}(\partial M_1)}{\text{Vol}(M_1)}$.

Isoperimetric inequality: Isoperimetric constant

$$\frac{\text{Area}(\partial A)}{\text{Vol}(A)^{(d-1)/d}} \geqslant \frac{\text{Area}(S_{d-1})}{\text{Vol}(B_d)^{(d-1)/d}}.$$

Replace Lebesgue meas by Gaussian one, ∞-dim.

- P. Lévy, 1919.
- M. Gromov, 1980, 1999.
- S. G. Bobkov, 1993. — & F. Götze 1999.
- D. Bakry and M. Ledoux, 1996.

2. A generalization

$(E, \mathscr{E}, \pi)$: prob. space, $\{(x,x): x \in E\} \in \mathscr{E} \times \mathscr{E}$.

$L^p(\pi)$, $\|\cdot\|_p$, $\|\cdot\| = \|\cdot\|_2$.

Symmetric form $(D, \mathscr{D}(D))$ on $L^2(\pi)$.

Integral operator.

$$D(f) = \frac{1}{2}\int_{E\times E} J(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2,$$

$$\mathscr{D}(D) = \{f \in L^2(\pi) : D(f) < \infty\},$$

$J \geqslant 0$: symmetric, no charge on $\{(x,x) : x \in E\}$.

Example. For a q-pair $(q(x), q(x, \mathrm{d}y))$ with π,

$$J(\mathrm{d}x, \mathrm{d}y) = \pi(\mathrm{d}x) q(x, \mathrm{d}y).$$

For a Q-matrix $Q = (q_{ij})$ with $(\pi_i > 0)$,

$$J_{ij} = \pi_i q_{ij} \ (j \neq i).$$

$$\lambda_1 := \inf\{D(f) : \pi(f) = 0,\ \|f\| = 1\}.$$

—spectral gap of $(D, \mathscr{D}(D))$,

$$\pi(f) = \int f \mathrm{d}\pi.$$

Theorem [G. F. Lawler & A. D. Sokal, 1988].

$$2k \geqslant \lambda_1 \geqslant \frac{1}{2M} k^2,$$

$$k := \inf_{\pi(A) \in (0,1)} \frac{J(A \times A^c)}{\pi(A) \wedge \pi(A^c)}.$$

$$\frac{\mathrm{d}J(\cdot, E)}{\mathrm{d}\pi} \leqslant M.$$ Bounded operators.

Six books:

- C. (1992), From Markov Chains to Non-Equilibrium Particle Systems, World Scientific, Singapore
- Sinclair, A. (1993), Algorithms for Random Generation and Counting: A Markov Chain Approach, Birkhäuser
- Colin de Verdière, Y. (1998), Spectres de Graphes, Publ. Soc. Math. France

- Chung, F. R. K. (1997), Spectral Graph Theory, CBMS, **92**, AMS, Providence, Rhode Island
- Saloff-Coste, L. (1997), Lectures on finite Markov chains, LNM **1665**, 301–413, Springer-Verlag
- Aldous, D. G. & Fill, J. A. (1994–), Reversible Markov Chains and Random Walks on Graphs

Basic inequalities

$\mathrm{Var}(f) = \pi(f^2) - \pi(f)^2.$

Poincaré inequality : $\mathrm{Var}(f) \leqslant CD(f), \ \lambda_1^{-1}.$

Log Sobolev : $\int f^2 \log\left(f^2/\|f\|^2\right) \mathrm{d}\pi \leqslant CD(f).$

Nash inequality : $\mathrm{Var}(f) \leqslant CD(f)^{1/p} \|f\|_1^{2/q}$

$$1/p + 1/q = 1$$

Equivalent if $\|f\|_1 \longrightarrow \|f\|_r$ for all $r \in (1, 2)$.

Liggett inequality : $\mathrm{Var}(f) \leqslant CD(f)^{1/p} \mathbf{Lip}(f)^{2/q}$

Liggett-Stroock inequality :

$$\mathrm{Var}(f) \leqslant CD(f)^{1/p} V(f)^{1/q},$$

$$V : L^2(\pi) \to [0, \infty], V(c_1 f + c_2) = c_1^2 V(f).$$

LogS: Differential operator. Integral operator.

Theorem [Diaconis & Saloff-Coste, 1996].

Let $\sum_j |q_{ij}| = 1$. Then

$$\mathrm{Ent}\,(f^2) \leqslant \frac{2}{\sigma} D(f),$$

$$\sigma \geqslant \frac{2(1-2\pi_*)\lambda_1}{\log[1/\pi_* - 1]}, \qquad \pi_* = \min_i \pi_i.$$

Can be sharp. Works only for finite state space.

Similar for Nash inequality.

3. New results

New forms of Cheeger's constants

Inequalities	Constant $k^{(\alpha)}$
Poincaré (C.&W.)	$\inf_{\pi(A)\in(0,1)} \dfrac{J^{(\alpha)}(A\times A^c)}{\pi(A)\wedge\pi(A^c)}$
Nash (C.)	$\inf_{\pi(A)\in(0,1)} \dfrac{J^{(\alpha)}(A\times A^c)}{[\pi(A)\wedge\pi(A^c)]^{(\nu-1)/\nu}}$ $\nu=2(q-1)$
LogS (Wang)	$\lim_{r\to 0}\inf_{\pi(A)\in(0,r]} \dfrac{J^{(\alpha)}(A\times A^c)}{\pi(A)\sqrt{\log[e+\pi(A)^{-1}]}}$
(C.)	$\lim_{\delta\to\infty}\inf_{\pi(A)>0} \dfrac{J^{(\alpha)}(A\times A^c)+\delta\pi(A)}{\pi(A)\sqrt{1-\log\pi(A)}}$

$$J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y) = I_{\{r(x,y)^\alpha > 0\}} \frac{J(\mathrm{d}x, \mathrm{d}y)}{r(x,y)^\alpha}$$

$r \geqslant 0$, symmetric. $\alpha \in [0, 1]$.

$J^{(1)}(\mathrm{d}x, E)/\pi(\mathrm{d}x) \leqslant 1, \quad \pi$-a.e.

Theorem 1. $k^{(1/2)} > 0 \Longrightarrow$ corresponding inequality holds.

Four papers:

C. and F. Y. Wang (1998), C. (1999, 2000), F. Y. Wang (2001).

Estimates $\lambda_1 \geqslant \dfrac{{k^{(1/2)}}^2}{1 + \sqrt{1 - {k^{(1)}}^2}}$. Can be sharp!

Corollary. $(E, \mathscr{E}, \pi)$. Let $j(x,y) \geqslant 0$ be symmetric function with $j(x,x)=0$ and

$$j(x) := \int_E j(x,y)\pi(\mathrm{d}y) < \infty \qquad \forall x \in E.$$

Then, for symmetric form generated by $J(\mathrm{d}x, \mathrm{d}y) := j(x,y)\pi(\mathrm{d}x)\pi(\mathrm{d}y)$, we have

$$\lambda_1 \geqslant \frac{1}{8} \inf_{x \neq y} \frac{j(x,y)^2}{j(x) \vee j(y)}.$$

- Jump processes, L^p. F. Wang and Y. H. Zhang (2000); Y. H. Mao (2001). Preprints.
- Diffusions. F. Y. Wang (1999), M. Röckner and F. Y. Wang (2001). Slower convergence.

4. Splitting technique and existence criterion

Neumann $\longrightarrow$ Dirichlet.

$$\lambda_1 = \inf\{D(f) : f \in \mathscr{D}(D), \pi(f) = 0, \pi(f^2) = 1\},$$
$$\lambda_0(A) = \inf\{D(f) : f \in \mathscr{D}(D), f|_{A^c} = 0, \pi(f^2) = 1\}.$$

Cheeger's splitting technique (Lawler&Sokal: bdd):

$$\lambda_1 \geqslant \inf_B\{\lambda_0(B) \vee \lambda_0(B^c)\}\Big[\geqslant \inf_{\pi(B)\in(0,1/2]} \lambda_0(B)\Big].$$

Theorem 2 [C. & Wang(2000), C. (2000)].

For above symmetric form or general Dirichlet form $\big(D, \mathscr{D}(D)\big)$, we have

$$\inf_{\pi(A)\in(0,1/2]} \lambda_0(A) \leqslant \lambda_1 \leqslant 2 \inf_{\pi(A)\in(0,1/2]} \lambda_0(A).$$

Proof. **(a) Let** $f \in \mathscr{D}(D)$, $f|_{A^c} = 0, \pi(f^2) = 1.$

$$\pi(f^2) - \pi(f)^2 = 1 - \pi\big(fI_A\big)^2 \geqslant 1 - \pi\big(f^2\big)\pi(A)$$
$$= 1 - \pi(A) = \pi\big(A^c\big).$$

$$\lambda_1 \leqslant \frac{D(f)}{\pi(f^2) - \pi(f)^2} \leqslant \frac{D(f)}{\pi\big(A^c\big)} \Longrightarrow \lambda_1 \leqslant \frac{\lambda_0(A)}{\pi\big(A^c\big)}.$$

$$\Longrightarrow \lambda_1 \leqslant \inf_{\pi(A)\in(0,1)} \min\left\{\frac{\lambda_0(A)}{\pi(A^c)}, \frac{\lambda_0(A^c)}{\pi(A)}\right\}$$
$$= \inf_{\pi(A)\in(0,1/2]} \min\left\{\frac{\lambda_0(A)}{\pi(A^c)}, \frac{\lambda_0(A^c)}{\pi(A)}\right\}$$
$$\leqslant \inf_{\pi(A)\in(0,1/2]} \lambda_0(A)/\pi(A^c)$$
$$\leqslant 2 \inf_{\pi(A)\in(0,1/2]} \lambda_0(A).$$

Works general $D(f)$**!**

(b) For $\varepsilon > 0$, choose f_ε: $\pi(f_\varepsilon) = 0$, $\pi(f_\varepsilon^2) = 1$, $\lambda_1 + \varepsilon \geqslant D(f_\varepsilon)$.
Choose c_ε: $\quad \pi(f_\varepsilon < c_\varepsilon), \quad \pi(f_\varepsilon > c_\varepsilon) \leqslant 1/2$.
Set $f_\varepsilon^\pm = (f_\varepsilon - c_\varepsilon)^\pm, \quad B_\varepsilon^\pm = \{f_\varepsilon^\pm > 0\}$.

$$D(f) = \frac{1}{2}\int J(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2.$$

$$\begin{aligned}\lambda_1 + \varepsilon \geqslant D(f_\varepsilon) &= D(f_\varepsilon - c_\varepsilon)\\ &= \frac{1}{2}\int J(\mathrm{d}x, \mathrm{d}y)\Big[\big|f_\varepsilon^+(y) - f_\varepsilon^+(x)\big|\\ &\qquad + \big|f_\varepsilon^-(y) - f_\varepsilon^-(x)\big|\Big]^2\end{aligned}$$

$$\lambda_1 + \varepsilon \geqslant \frac{1}{2}\int J(\mathrm{d}x, \mathrm{d}y)\big(f_\varepsilon^+(y) - f_\varepsilon^+(x)\big)^2$$
$$+ \frac{1}{2}\int J(\mathrm{d}x, \mathrm{d}y)\big(f_\varepsilon^-(y) - f_\varepsilon^-(x)\big)^2$$
$$\geqslant \lambda_0\big(B_\varepsilon^+\big)\pi\big(\big(f_\varepsilon^+\big)^2\big) + \lambda_0\big(B_\varepsilon^-\big)\pi\big(\big(f_\varepsilon^-\big)^2\big)$$
$$\geqslant \inf_{\pi(B)\in(0,1/2]} \lambda_0(B)\pi\big(\big(f_\varepsilon^+\big)^2 + \big(f_\varepsilon^-\big)^2\big)$$
$$= (1 + c_\varepsilon^2) \inf_{\pi(B)\in(0,1/2]} \lambda_0(B) \geqslant \inf_{\pi(B)\in(0,1/2]} \lambda_0(B).$$

ε arbitrary! Proof is done for the first case!

(c) For general Dirichlet form:

$$D(f) = \lim_{t\downarrow 0} \frac{1}{2t}\int \pi(\mathrm{d}x)P_t(x, \mathrm{d}y)[f(y) - f(x)]^2.$$

Applications to Neumann eigenvalue

Theorem [V. G. Maz'ya, 1973; Z. Vondraček, 1996; M. Fukushima and T. Uemura 2002].
For regular transient Dirichlet form on A,

$$(4\Theta(A))^{-1} \leqslant \lambda_0(A) \leqslant \Theta(A)^{-1},$$

$$\Theta(A) = \sup_{\text{compact } K\subset A} \frac{\pi(K)}{\text{Cap}(K)}.$$

Theorem. For regular Dirichlet form, we have

$$\inf_{\pi(A)\in(0,1/2]} (4\Theta(A))^{-1} \leqslant \lambda_1 \leqslant 2 \inf_{\pi(A)\in(0,1/2]} \Theta(A)^{-1}.$$

In particular, $\lambda_1 > 0$ iff $\sup_{\pi(A)\in(0,1/2]} \Theta(A) < \infty$.

Informal criterion [C., F. Y. Wang, 2000].

$\lambda_1 > 0 \iff \exists$ compact A such that $\lambda_0(A^c) > 0$,

$$\lambda_0(A^c) = \inf\{D(f,f) : \ f|_A = 0, \pi(f^2) = 1\}.$$

Criterion [C., F. Y. Wang, 2000].

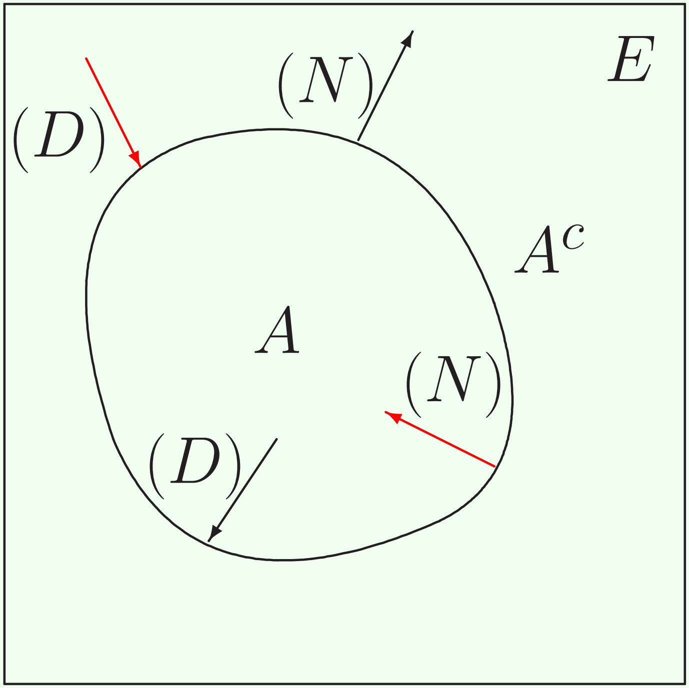
Let $A \subset B$, $0 < \pi(A)$, $\pi(B) < 1$. Then

$$\frac{\lambda_0(A^c)}{\pi(A)} \geqslant \lambda_1 \geqslant \frac{\lambda_1(B)[\lambda_0(A^c)\pi(B) - 2M_A\pi(B^c)]}{2\lambda_1(B) + \pi(B)^2[\lambda_0(A^c) + 2M_A]},$$

where $M_A = \text{ess sup}_{A,\,\pi} J(\mathrm{d}x, A^c)/\pi(\mathrm{d}x)$.

In particular, $\lambda_1 > 0 \iff \lambda_0(A^c) > 0$.

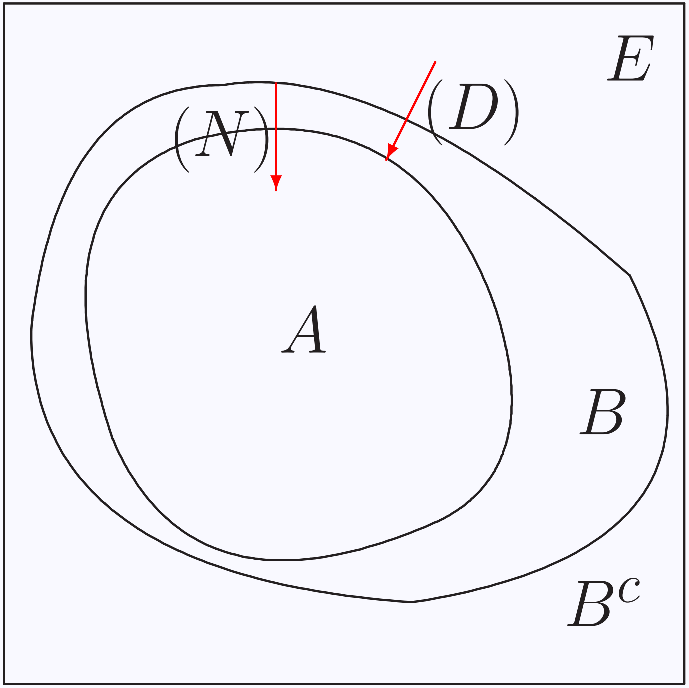
Four choices of boundary condition



(D): Dirichlet boundary

(N): Neumann boundary

Non-local, need intersection.



5. Sketch of the proofs of Theorem 1

Theorem 1. $k^{(1/2)} > 0 \Longrightarrow$ corresponding inequality holds.

(a) Local form

Given $B \subset E$, $\pi(B) > 0$. For $f|_{B^c} = 0$,

$$D^{(\alpha)}(f) = \frac{1}{2}\int_{B\times B} J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2 + \int_B J^{(\alpha)}(\mathrm{d}x, B^c) f(x)^2 =: \widetilde{D}_B^{(\alpha)}(f).$$

$$\widetilde{D}_B^{(\alpha)}(f) = \widetilde{D}_B^{(\alpha)}(f I_B).$$

$$\begin{aligned}\lambda_0(B) &:= \inf\{D(f) : f|_{B^c} = 0,\ \|f\| = 1\} \\ &= \inf\left\{\widetilde{D}_B(f) :\ \pi(f^2 I_B) = 1\right\}.\end{aligned}$$

Set

$$\begin{aligned}h_B^{(\alpha)} &= \inf_{A\subset B,\, \pi(A)>0} \frac{J^{(\alpha)}(A\times(B\setminus A))+J^{(\alpha)}(A\times B^c)}{\pi(A)} \\ &= \inf_{A\subset B,\, \pi(A)>0} \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)}.\end{aligned}$$

Ignoring B. $\left(D^{(\alpha)},\ \mathscr{D}(D^{(\alpha)})\right),\ \ \lambda_0,\ \ h^{(\alpha)}.$

(b) Dirichlet eigenvalue, with killing

$$D^{(\alpha)}(f) = \frac{1}{2}\int J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2$$
$$+ \int K^{(\alpha)}(\mathrm{d}x) f(x)^2$$

$$K^{(\alpha)}(\mathrm{d}x) = K(\mathrm{d}x)/s(x)^{\alpha}.$$

$$\lambda_0 = \inf\left\{D(f) : \pi\left(f^2\right) = 1\right\}.$$

$$h^{(\alpha)} = \inf_{\pi(A)>0} \frac{J^{(\alpha)}(A \times A^c) + K^{(\alpha)}(A)}{\pi(A)}.$$

$$[J^{(1)}(\mathrm{d}x, E) + K^{(1)}(\mathrm{d}x)]/\pi(\mathrm{d}x) \leqslant 1, \quad \pi\text{-a.s}$$

Theorem [C. & Wang(2000)].

$$\lambda_0 \geqslant \frac{{h^{(1/2)}}^2}{1 + \sqrt{1 - {h^{(1)}}^2}}, \quad \lambda_1 \geqslant \frac{{k^{(1/2)}}^2}{1 + \sqrt{1 - {k^{(1)}}^2}}.$$

(c) Ignore killing: "compactification"

$$h^{(\alpha)} = \inf\left\{\frac{1}{2}\int J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y)|f(x) - f(y)| + K^{(\alpha)}(f) :\right.$$

$$\left. f \geqslant 0,\ \pi(f) = 1\right\}.$$

$$= \inf\left\{\frac{1}{2}\int_{E^* \times E^*} J^{*(\alpha)}(\mathrm{d}x, \mathrm{d}y)|f^*(x) - f^*(y)| :\right.$$

$$\left. f \geqslant 0,\ \pi(f) = 1\right\},$$

$E^* = E \cup \{\infty\}$, $J \to J^*$, $J^{(\alpha)} \to J^{*(\alpha)}$,
$D^{(\alpha)} \to D^{*(\alpha)}$, $f \to f^* := f I_E$.

Omit $(*)$ for simplicity!

For every f: $\pi(f^2) = \|f\|^2 = 1$.

$$
\begin{aligned}
h^{(1)^2} &\leqslant \left\{\frac{1}{2}\int J^{(1)}(\mathrm{d}x, \mathrm{d}y)\big|f(y)^2 - f(x)^2\big|\right\}^2 \\
&\leqslant \frac{1}{2}D^{(1)}(f)\int J^{(1)}(\mathrm{d}x, \mathrm{d}y)\big[f(y) + f(x)\big]^2 \\
&= \frac{1}{2}D^{(1)}(f)\left\{2\int J^{(1)}(\mathrm{d}x, \mathrm{d}y)\big[f(y)^2 + f(x)^2\big] \right. \\
&\qquad \left. - \int J^{(1)}(\mathrm{d}x, \mathrm{d}y)\big[f(y) - f(x)\big]^2\right\} \\
&\leqslant D^{(1)}(f)\big[2 - D^{(1)}(f)\big].
\end{aligned}
$$

$$
\Longrightarrow D^{(1)}(f) \geqslant 1 - \sqrt{1 - h^{(1)^2}}.
$$

$$
\begin{aligned}
{h^{(1/2)}}^2 &\leqslant \left\{\frac{1}{2}\int J^{(1/2)}(\mathbf{d}x, \mathbf{d}y)\big|f(y)^2 - f(x)^2\big|\right\}^2 \\
&= \left\{\frac{1}{2}\int J(\mathbf{d}x, \mathbf{d}y)|f(y) - f(x)|\right. \\
&\qquad\qquad \left.\cdot\, I_{\{r(x,y)>0\}}\frac{|f(y) + f(x)|}{\sqrt{r(x,y)}}\right\}^2 \\
&\leqslant \frac{1}{2}D(f)\int J^{(1)}(\mathbf{d}x, \mathbf{d}y)\big[f(y) + f(x)\big]^2
\end{aligned}
$$

(Good use of Schwarz inequality!)

$$
\begin{aligned}
&\leqslant D(f)\big[2 - D^{(1)}(f)\big]. \\
{h^{(1/2)}}^2 &\leqslant D(f)\left[1 + \sqrt{1 - {h^{(1)}}^2}\right].
\end{aligned}
$$

(d) Final step: Return to Neumann

$$\lambda_1 \geqslant \inf_{\pi(B)\leqslant 1/2} \lambda_0(B) \geqslant \inf_{\pi(B)\leqslant 1/2} \frac{{h_B^{(1/2)}}^2}{1+\sqrt{1-{h_B^{(1)}}^2}}$$

$$\geqslant \inf_{\pi(B)\leqslant 1/2} \frac{\inf_{\pi(B)\leqslant 1/2} {h_B^{(1/2)}}^2}{1+\sqrt{1-{h_B^{(1)}}^2}}$$

$$\geqslant \frac{\inf_{\pi(B)\leqslant 1/2} {h_B^{(1/2)}}^2}{1+\sqrt{1-\inf_{\pi(B)\leqslant 1/2} {h_B^{(1)}}^2}} = \frac{{k^{(1/2)}}^2}{1+\sqrt{1-{k^{(1)}}^2}}.$$

Ten Explicit Criteria of One-dimensional Processes

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Stochastic analysis on large scale interacting systems

Shonan Village Center, Hayama, Japan

(July 17–26, 2002)[26]

The First Sino-German Conference on Stochastic Analysis—A Satellite Conference of ICM 2002

The Sino-German Center, Beijing

(August 29–September 3, 2002)[29]

Contents

1. Three traditional types of ergodicity

Regular Q-matrix: $q_{ij} \geqslant 0, i \neq j,\ q_i = -q_{ii} = \sum_{j\neq i} q_{ij} < \infty$. Unique. Stationary distri.(π_i).

Ordinary ergodicity : $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0$

Exponential erg. : $\lim_{t\to\infty} e^{\hat\alpha t}|p_{ij}(t) - \pi_j| = 0$

Strong ergodicity : $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$

$$\iff \lim_{t\to\infty} e^{\hat\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0$$

Pointwise $\longrightarrow \|p_t(x, \cdot) - \pi\|_{\text{Var}}$

Classical Theorem :

Strong erg. $\Longrightarrow$ Exponential erg. $\Longrightarrow$ Ordinary erg.

Criteria (1953–1981).

Let $H \neq \emptyset$, finite.

- Ordinary Ergodic iff

$$\begin{cases} \sum_j q_{ij} y_j \leqslant -1, & i \notin H \\ \sum_{i\in H} \sum_{j\neq i} q_{ij} y_j < \infty \end{cases} \qquad (*)$$

has finite solution $(y_i \geqslant 0)$.

- Exp. ergodic iff for some $0 < \lambda < q_i$ for all i,

$$\begin{cases} \sum_j q_{ij} y_j \leqslant -\lambda y_i - 1, & i \notin H \\ \sum_{i\in H} \sum_{j\neq i} q_{ij} y_j < \infty \end{cases}$$

has finite solution $(y_i \geqslant 0)$.

- Strong ergodic iff $(*)$ has bdd solution $(y_i \geqslant 0)$.

Satisfactory!? Look for explicit criterion!

Birth-death processes

$$\begin{aligned} i &\longrightarrow i+1 \quad \text{at rate} \quad b_i = q_{i,i+1} > 0 \\ &\longrightarrow i-1 \quad \text{at rate} \quad a_i = q_{i,i-1} > 0. \end{aligned}$$

$$\mu_0 = 1, \qquad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n}, \quad n \geqslant 1.$$

Theorem [R.L.Tweedie(1981)]: Explicit!

$$S \risingdotseq \sum_{n\geqslant 1}\mu_n \sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty \Longrightarrow \text{Exp. ergodicity}$$

Theorem [H.Z.Zhang, X.Lin and Z.T.Hou 2000)]:

$$S<\infty \Longleftrightarrow \text{Strong ergodicity.}$$

Not enough for exponentially or strongly ergodic convergence rate.

Criterion for exponential ergodicity?

E. Van Doorn (1981—. One book, # papers).

Works only on birth-death processes!

Upper and lower bounds:

$$\hat{\alpha} \geqslant \inf_{i \geqslant 0} \left\{ a_{i+1} + b_i - \sqrt{a_i b_i} - \sqrt{a_{i+1} b_{i+1}} \right\}$$

Exact when a_i & b_i are constant.

Formula for lower bounds (implicated)(C. 96):

$$\hat{\alpha} = \sup_{v>0} \inf_{i \geqslant 0} \{ a_{i+1} + b_i - a_i / v_{i-1} - b_{i+1} v_i \}$$

Based on Karlin-Mcgregor's representation thm.

No criterion!

2. The first (non-trivial) eigenvalue (spectral gap)

Birth-death processes. Reversible: $\mu_i q_{ij} = \mu_j q_{ji}$. $Z = \sum_i \mu_i$, $\pi_i = \mu_i/Z$. $L^2(\pi)$. $\|\cdot\|$.

$$\|P_t f - \pi(f)\| \leqslant \|f - \pi(f)\| e^{-\lambda_1 t}, \quad t \geqslant 0$$

$\pi(f) = \int f \mathrm{d}\pi$, λ_1: first eigenvalue of $(-Q)$.

Lower bounds:

- W. G. Sullivan(1984):

Theorem. $c_1 \geqslant \dfrac{\sum_{j\geqslant i}\mu_j}{\mu_i}$, $c_2 \geqslant \dfrac{\mu_{i+1}}{\mu_i b_i}$, $\lambda_1 \geqslant \dfrac{1}{4c_1^2 c_2}$.

- T. M. Liggett(1989), Particle Systems, ∞-dim.

Theorem. $c_1 \geqslant \dfrac{\sum_{j\geqslant i}\mu_j}{\mu_i a_i}$, $\quad c_2 \geqslant \dfrac{\sum_{j\geqslant i}\mu_j a_j}{\mu_i a_i}$,

$\lambda_1 \geqslant \dfrac{1}{4c_1c_2}$. Exact when a_i & b_i are constant.

Theorem. Bdd $a_i\&b_i$, $\lambda_1 > 0$ iff (μ_i) has exp. tail.

- C. Landim, S. Sethuraman, S. R. S. Varadhan (1996)

- $\cdots\cdots$

Definition. Classical variational formula:

$$\lambda_1 = \inf\Big\{D(f) : \pi(f) = 0, \pi\big(f^2\big) = 1\Big\},$$

$$D(f) = \frac{1}{2}\sum_{i,j}\pi_i q_{ij}(f_j - f_i)^2,$$

$$\mathscr{D}(D) = \{f \in L^2(\pi) : D(f) < \infty\},$$

$\pi(f) = \int f \mathrm{d}\pi.$ **Upper bounds:** Easier!

3. One-dimensional diffusions

$L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$, $[0, \infty)$, $a(x) > 0$.

Sturm-Liouville eigenvalue. Long history!

ODE. Spectral Theory.

Book: Y.Egorov & V.Kondratiev (1996), On Spectral Theory of Elliptic Operators, Birkbäuser.

Theorem. Let $b(x) \equiv 0$, $\delta = \sup_{x>0} x \int_x^\infty a^{-1}$.

- I. S. Kac & M. G. Krein (1958):
  $\delta^{-1} \geqslant \lambda_0 \geqslant (4\delta)^{-1}$, $\quad \lambda_0 \longrightarrow f(0) = 0$.
- S. Kotani & S. Watanabe (1982):
  $\delta^{-1} \geqslant \lambda_1 \geqslant (4\delta)^{-1}$.

Poincaré inequalities

$$\lambda_1: \quad \mathrm{Var}(f) = \|f - \pi(f)\|^2 \leqslant \lambda_1^{-1} D(f)$$

$$\lambda_0: \quad \|f\|^2 \leqslant \lambda_0^{-1} D(f), \quad f(0) = 0.$$

4. Weighted Hardy inequality.

$$\int_0^\infty \left(\frac{f}{x}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f'^p, \; f(0)=0, f' \geqslant 0,$$

G.H.Hardy (1920). For a long period, great efforts.

Weighted Hardy inequality.

$$\int_0^\infty f^2 \mathrm{d}\nu \leqslant A \int_0^\infty f'^2 \mathrm{d}\lambda, \quad f \in C^1, f(0) = 0,$$

$\nu, \lambda \geqslant 0$ Borel measures.

Harmonic analysis. Books:

- Opic, B. & Kufner, A. (1990), Hardy-type Inequalities, Longman, New York.
- Dynkin, E. M. (1990), EMS, Springer-Valerg (1991), Berlin.
- Mazya, V. G. (1985), Sobolev Spaces, Springer-Valerg.

Survey article:

- Davies, E. B. (1999), A review of Hardy inequality, Operator Theory: Adv. & Appl. 110, 55–67.

Difficulties

Example 1: Trivial case(two points). Two parameters.

$$\begin{pmatrix} -b & b \\ a & -a \end{pmatrix}, \quad \lambda_1 = a + b.$$

λ_1 is increasing in each of the parameters!

Example 2: Three points. Four parameters.

$$\begin{pmatrix} -b_0 & b_0 & 0 \\ a_1 & -(a_1 + b_1) & b_1 \\ 0 & a_2 & -a_2 \end{pmatrix},$$

$$\lambda_1 = 2^{-1}\Big[a_1 + a_2 + b_0 + b_1 - \sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1 b_1}\Big].$$

Example 3: Four points.

Six parameters: $b_0, b_1, b_2, a_1, a_2, a_3$.

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3}\,\left(3\,B - D^2\right)}{3\,C},$$

where

$$D = a_1 + a_2 + a_3 + b_0 + b_1 + b_2,$$

$$B = a_3\,b_0 + a_2\,(a_3 + b_0) + a_3\,b_1 + b_0\,b_1 + b_0\,b_2 + b_1\,b_2 + a_1\,(a_2 + a_3 + b_2)\,,$$

$$C = \left(A + \sqrt{4\left(3\,B - D^2\right)^3 + A^2}\right)^{1/3},$$

$$A = -2\,a_1^3 - 2\,a_2^3 - 2\,a_3^3 + 3\,a_3^2\,b_0 + 3\,a_3\,b_0^2 - 2\,b_0^3 + 3\,a_3^2\,b_1 - 12\,a_3\,b_0\,b_1 + 3\,b_0^2\,b_1 + 3\,a_3\,b_1^2 + 3\,b_0\,b_1^2 - 2\,b_1^3 - 6\,a_3^2\,b_2 + 6\,a_3\,b_0\,b_2 + 3\,b_0^2\,b_2 + 6\,a_3\,b_1\,b_2 - 12\,b_0\,b_1\,b_2 + 3\,b_1^2\,b_2 - 6\,a_3\,b_2^2 + 3\,b_0\,b_2^2 + 3\,b_1\,b_2^2 - 2\,b_2^3 + 3\,a_1^2\,(a_2 + a_3 - 2\,b_0 - 2\,b_1 + b_2) + 3\,a_2^2\,[a_3 + b_0 - 2\,(b_1 + b_2)] + 3\,a_2\left[a_3^2 + b_0^2 - 2\,b_1^2 - b_1\,b_2 - 2\,b_2^2 - a_3(4\,b_0 - 2\,b_1 + b_2) + 2\,b_0(b_1 + b_2)\right] + 3\,a_1\left[a_2^2 + a_3^2 - 2\,b_0^2 - b_0\,b_1 - 2\,b_1^2 - a_2(4\,a_3 - 2\,b_0 + b_1 - 2\,b_2) + 2\,b_0\,b_2 + 2\,b_1\,b_2 + b_2^2 + 2\,a_3(b_0 + b_1 + b_2)\right].$$

The role of each parameter is completely mazed!

Not solvable when state has more than five points!

Conclusion: Impossible to compute λ_1 explicitly!

Perturbation of eigenvalues

Example: (Infinite) Birth-death processes.

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$		
$i+1$	$2i+\left(4+\sqrt{2}\,\right)$		

Perturbation of eigenvalues

Example: (Infinite) Birth-death processes.

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	
$i+1$	$2i+\left(4+\sqrt{2}\right)$	3	

Perturbation of eigenvalues

Example: (Infinite) Birth-death processes.

$b_i\ (i \geqslant 0)$	$a_i\ (i \geqslant 1)$	λ_1	degree of eigenfun.
$i+\beta$ $(\beta > 0)$	$2i$	1	1
$i+1$	$2i+3$	2	2
$i+1$	$2i+\left(4+\sqrt{2}\right)$	3	3

In general, it is too hard to estimate λ_1!

5. One main result. Dual variational formulas and explicit bounds for birth-death processes.

Theorem (C.). (1). $\hat{\alpha} = \lambda_1$ (1991).

(2). $$\lambda_1 = \sup_{w\in\mathscr{W}} \inf_{i\geqslant 0} I_i(w)^{-1} \quad (1996),$$

$$= \inf_{w\in\mathscr{W}'} \sup_{i\geqslant 0} I_i(w)^{-1} \quad (2001),$$

(3). $Z\delta^{-1} \geqslant {\delta'_n}^{-1} \geqslant \lambda_1 \geqslant \delta_n^{-1} \geqslant (4\delta)^{-1}$ (2000)

$$\mathscr{W} = \left\{w : w_i \uparrow\uparrow, \sum_i \mu_i w_i \geqslant 0\right\},$$

$$I_i(w) = \frac{1}{\mu_i b_i(w_{i+1}-w_i)} \sum_{j\geqslant i+1} \mu_j w_j, \quad w\in\mathscr{W}$$

$$Z = \sum_i \mu_i, \quad \delta = \sup_{i>0} \sum_{j\leqslant i-1} \frac{1}{\mu_j b_j} \sum_{j\geqslant i} \mu_j$$

$$\lambda_1 = \inf\left\{D(f) : \pi(f) = 0, \pi\left(f^2\right) = 1\right\}$$

Theorem [C. (99, 00, 01), C. & F. Y. Wang(97)]

(1). $\lambda_0 \geqslant \sup_{f\in\mathscr{F}} \inf_{x>0} I(f)(x)^{-1}$

$$\lambda_0 \leqslant \inf_{f\in\mathscr{F}'} \sup_{x>0} I(f)(x)^{-1}$$

(2). $\delta^{-1} \geqslant {\delta_n'}^{-1} \downarrow \geqslant \lambda_0 \geqslant \delta_n^{-1} \uparrow \geqslant (4\delta)^{-1}$.

(3). λ_0 (resp. λ_1) > 0 iff $\delta < \infty$.

$$I(f)(x) = \frac{e^{-C(x)}}{f'(x)} \int_x^\infty f e^C/a, \quad C(x) = \int_0^x b/a$$

$$\mathscr{F} = \{f \in L^1(\pi) : \pi(f) \geqslant 0 \text{ and } f'|_{(0,\infty)} > 0\}$$

$$\delta = \sup_{x>0} \int_0^x e^{-C} \int_x^\infty e^C/a$$

Theorem [B. Muckenhoupt (1972)].

$$\int_0^\infty f^2 \mathrm{d}\nu \leqslant A \int_0^\infty f'^2 \mathrm{d}\lambda, \quad f \in C^1, f(0) = 0,$$

$$B \leqslant A \leqslant 4B.$$

$$B = \sup_{x>0} \nu[x, \infty) \int_x^\infty \frac{1}{\mathrm{d}\lambda_{\text{abs}}/\mathrm{dLeb}}.$$

Setting $\nu = \pi, \ \lambda = e^C \mathbf{d}x,$

$$\delta = \sup_{x>0} \int_0^x e^{-C} \int_x^\infty e^C / a.$$

Muckenhoupt's Theorem :

$$\Longrightarrow \delta \leqslant A = \lambda_0^{-1} \leqslant 4\delta.$$

5'. A piece of the proofs

Proof of the explicit lower bound of λ_0

$\varphi(x) := \int_0^x e^C.$ Integration by parts formula:

$$\int_x^D \frac{\sqrt{\varphi}\, e^C}{a} = -\int_x^D \sqrt{\varphi}\, \mathrm{d}\Big(\int_\cdot^D \frac{e^C}{a}\Big) \leqslant \frac{2\delta}{\sqrt{\varphi(x)}}.$$

$$I\big(\sqrt{\varphi}\,\big)(x) = \frac{e^{-C(x)}}{\big(\sqrt{\varphi}\big)'(x)} \int_x^D \frac{\sqrt{\varphi} e^C}{a}$$

$$\leqslant \frac{e^{-C(x)}\sqrt{\varphi(x)}}{(1/2)e^{-C(x)}} \cdot \frac{2\delta}{\sqrt{\varphi(x)}}$$

$$= 4\delta. \qquad \delta = \sup_{x>0} \int_0^x e^{-C} \int_x^\infty e^C / a.$$

For Laplacian on compact Riemannian manifolds,

$$\lambda_1 \geqslant \sup_{f\in\mathscr{F}} \inf_{r\in(0,D)} I(f)(r)^{-1} =: \xi_1$$

$$\delta^{-1} \geqslant {\delta_n'}^{-1} \downarrow\geqslant \xi_1 \geqslant \delta_n^{-1} \uparrow\geqslant (4\delta)^{-1}.$$

Contain none of the sharp estimates.

Determine the exact value of ξ_1:

Theorem [C., E. Scacciatelli, L. Yao (2001)]

$$\xi_1 \geqslant \frac{\pi^2}{D^2} + \frac{K}{2}.$$

D: diameter, K: lower bound of Ricci curvature.

6. Basic inequalities

Poincaré inequality : $\mathrm{Var}(f) \leqslant \lambda_1^{-1} D(f)$

LogS inequality : $\mathrm{Ent}\big(f^2\big) \leqslant 2\sigma^{-1} D(f)$

$$\text{where } \mathrm{Ent}(f) = \int f \log \frac{f}{\pi(f)} \mathrm{d}\pi, \qquad f \geqslant 0.$$

Nash inequality : $\mathrm{Var}(f)^{1+2/\nu} \leqslant \eta^{-1} D(f) \|f\|_1^{4/\nu}$ (for some $\nu > 0$).

Poincaré inequality $\iff \mathrm{Var}(P_t f) \leqslant \mathrm{Var}(f)\, e^{-2\lambda_1 t}$.

LogS inequality $\implies \mathrm{Ent}(P_t f) \leqslant \mathrm{Ent}(f)\, e^{-2\sigma t}$,

Nash inequality $\iff \mathrm{Var}(P_t f) \leqslant C \|f\|_1^2\, t^{-\nu}$.

7. Ten criteria for birth-death processes

Birth-death processes

$$i \to i+1 \quad \text{at rate} \quad b_i = q_{i,i+1} > 0$$
$$\to i-1 \quad \text{at rate} \quad a_i = q_{i,i-1} > 0.$$

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n}, \ n \geqslant 1;$$
$$\mu[i,k] = \sum_{i \leqslant j \leqslant k} \mu_j.$$

Property	Criterion
Uniqueness	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[0,n]=\infty \quad (*)$
Recurrence	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}=\infty$
Ergodicity	$(*)$ & $\mu[0,\infty)<\infty$
Exp. ergodicity Poincaré inequality	$(*)$ & $\sup_{n\geqslant 1}\mu[n,\infty)\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Discrete spectrum	$(*)$ & $\lim_{n\to\infty}\sup_{k\geqslant n+1}\mu[k,\infty)\sum_{j=n}^{k-1}\frac{1}{\mu_j b_j}=0$
LogS inequality	$(*)$ & $\sup_{n\geqslant 1}\mu[n,\infty)\log[\mu[n,\infty)^{-1}]\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Strong ergodicity L^1-exp. converg.	$(*)$ & $\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[n+1,\infty)=\sum_{n\geqslant 1}\mu_n\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Nash inequality	$(*)$ & $\sup_{n\geqslant 1}\mu[n,\infty)^{(\nu-2)/\nu}\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty(\varepsilon)$

$(*)$ &. $\quad(\varepsilon)$.

Property	Criterion
Uniq.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[0,n]=\infty \quad (*)$
Recur.	$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}=\infty$
Ergod.	$(*)$ & $\mu[0,\infty)<\infty$
Exp. erg. Poinc.	$(*)$& $\sup_{n\geqslant 1}\mu[n,\infty)\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Dis. sp.	$(*)$& $\lim_{n\to\infty}\sup_{k\geqslant n+1}\mu[k,\infty)\sum_{j=n}^{k-1}\frac{1}{\mu_j b_j}=0$
LogS	$(*)$& $\sup_{n\geqslant 1}\mu[n,\infty)\log[\mu[n,\infty)^{-1}]\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Str. erg. L^1-exp.	$(*)$& $\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}\mu[n+1,\infty)=\sum_{n\geqslant 1}\mu_n\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty$
Nash	$(*)$& $\sup_{n\geqslant 1}\mu[n,\infty)^{(\nu-2)/\nu}\sum_{j\leqslant n-1}\frac{1}{\mu_j b_j}<\infty(\varepsilon)$

8. Eleven criteria for one-dimensional diffusions

Diffusion processes on $[0, \infty)$

Reflecting boundary at 0.

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}. \quad C(x) = \int_0^x b/a,$$

$$\mathrm{d}\mu = e^C \mathrm{d}x/a, \quad \mu[x, y] = \int_x^y e^C/a.$$

Property	Criterion
Uniqueness	$\int_0^\infty \mu[0,x]e^{-C(x)} = \infty \quad (*)$
Recurrence	$\int_0^\infty e^{-C(x)} = \infty$
Ergodicity	$(*)\ \&\ \mu[0,\infty) < \infty$
Exp. ergodicity Poincaré inequality	$(*)\ \&\ \sup_{x>0} \mu[x,\infty) \int_0^x e^{-C} < \infty$
Discrete spectrum	$(*)\ \&\ \lim_{n\to\infty} \sup_{x>n} \mu[x,\infty) \int_n^x e^{-C} = 0$
LogS inequality Exp. conv. in entropy	$(*)\ \&\ \sup_{x>0} \mu[x,\infty)\log[\mu[x,\infty)^{-1}]\int_0^x e^{-C} < \infty$
Strong ergodicity L^1-exp. conv.	$(*)\ \&\ \int_0^\infty \mu[x,\infty)e^{-C(x)} < \infty$
Nash inequality	$(*)\ \&\ \sup_{x>0} \mu[x,\infty)^{(\nu-2)/\nu}\int_0^x e^{-C} < \infty(\varepsilon)$

$(*)$ &. $\quad (\varepsilon)$.

Property	Criterion
Uniq.	$\int_0^\infty \mu[0,x]e^{-C(x)} = \infty \quad (*)$
Recur.	$\int_0^\infty e^{-C(x)} = \infty$
Ergod.	$(*)$ & $\mu[0,\infty) < \infty$
Exp. Erg. Poinc.	$(*)$ & $\sup_{x>0} \mu[x,\infty) \int_0^x e^{-C} < \infty$
Dis. Sp.	$(*)$ & $\lim_{n\to\infty} \sup_{x>n} \mu[x,\infty) \int_n^x e^{-C} = 0$
LogS Exp. ent.	$(*)$ & $\sup_{x>0} \mu[x,\infty)\log[\mu[x,\infty)^{-1}]\int_0^x e^{-C} < \infty$
Str. Erg. L^1-Exp.	$(*)$ & $\int_0^\infty \mu[x,\infty)e^{-C(x)} < \infty$
Nash	$(*)$ & $\sup_{x>0} \mu[x,\infty)^{(\nu-2)/\nu} \int_0^x e^{-C} < \infty(\varepsilon)$

Contributors:

- Bobkov, S. G. and Götze, F. (1999a, b)
- C. (1991, 1996, 2000, 2001)
- Miclo, L. (1999, 2000)
- Mao, Y. H. (2001, 2002a, b)
- Wang, F. Y. (2001)
- Zhang, H. Z., Lin, X. and Hou, Z. T. (2000)
- Zhang, Y. H. (2001)
- $\cdots\cdots$

9. New picture of ergodic theory

Theorem. For rever. Markov proc. with densities,

Nash inequality

$\swarrow\!\!\swarrow$ $\searrow\!\!\searrow$

LogS inequality $\qquad$ L^1-exp. convergence

$\Downarrow$ $\qquad$ $\|$

Exp. conv. in entropy $\qquad$ π-a.s. Strong ergodicity

$\Downarrow$ $\qquad$ $\Downarrow$

Poincaré inequality $\Longleftrightarrow$ π-a.s. Exp. ergodicity

$\Downarrow$

L^2-algebraic ergodicity

$\Downarrow$

Ordinary ergodicity

10. Go to Banach spaces

Theorem [Varopoulos, N. (1985); Carlen, E. A., Kusuoka, S., Stroock, D. W. (1987); Bakry, D., Coulhon, T., Ledoux, M. and Saloff-Coste, L. (1995)]. When $\nu > 2$, Nash inequality:

$$\|f - \pi(f)\|^{2+4/\nu} \leqslant C_1 D(f) \|f\|_1^{4/\nu}$$

is equivalent to the Sobolev-type inequality:

$$\|f - \pi(f)\|^2_{\nu/(\nu-2)} \leqslant C_2 D(f),$$

where $\|\cdot\|_p$ is the $L^p(\mu)$-norm.
This leads to general Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$:

$$\|(f - \pi(f))^2\|_{\mathbb{B}} \leqslant A D(f).$$

Poincaré-type inequality in Banach spaces:

$$\|f^2\|_{\mathbb{B}} \leqslant A_{\mathbb{B}} D(f), \quad f(0)=0,\ f\in \mathbb{C}_{\mathrm{d}}[0,D].$$

Theorem. (1) Variational formula for upper bounds

$$A_{\mathbb{B}} \leqslant \inf_{f\in\mathscr{F}} \sup_{x\in(0,D)} \frac{e^{-C(x)}}{f'(x)} \|f I_{(x,D)}\|_{\mathbb{B}},\ C(x)=\int_0^x \frac{b}{a}$$

$$\mathscr{F} = \{f\in C[0,D] : f(0)=0,\ f'|_{(0,D)}>0\}.$$

(2) Explicit bounds and criterion.

$$B_{\mathbb{B}} \leqslant C_{\mathbb{B}} \leqslant A_{\mathbb{B}} \leqslant D_{\mathbb{B}}(n)\downarrow \leqslant 4B_{\mathbb{B}}.$$

$$B_{\mathbb{B}} = \sup_{x\in(0,D)} \varphi(x)\|I_{(x,D)}\|_{\mathbb{B}},\ \varphi(x)=\int_0^x e^{-C}$$

Back to Bobkov & Götze, JFA 1999.

Variational Formulas of Poincaré-type Inequalities for One-dimensional Processes

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(July 29–30, 2002)[29]

Contents

1. Introduction.

Shonan Village Center

Diffusion processes on $[0, \infty)$

Reflecting boundary at 0.

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}. \quad C(x) = \int_0^x b/a,$$

$$\mathrm{d}\mu/\mathrm{d}x = e^C/a, \quad \mu[x,y] = \int_x^y e^C/a.$$

$$Z = \mu(0,\infty), \quad \mathrm{d}\pi = \mathrm{d}\mu/Z.$$

Eleven criteria for one-dimensional diffusions

Property	Criterion
Uniqueness	$\int_0^\infty \mu[0,x]e^{-C(x)} = \infty \quad (*)$
Recurrence	$\int_0^\infty e^{-C(x)} = \infty$
Ergodicity	$(*)$ & $\mu[0,\infty) < \infty$
Exp. ergodicity Poincaré inequality	$(*)$ & $\sup_{x>0} \mu[x,\infty) \int_0^x e^{-C} < \infty$
Discrete spectrum	$(*)$ & $\lim_{n\to\infty} \sup_{x>n} \mu[x,\infty) \int_n^x e^{-C} = 0$
LogS inequality Exp. conv. in entropy	$(*)$ & $\sup_{x>0} \mu[x,\infty)\log[\mu[x,\infty)^{-1}] \int_0^x e^{-C} < \infty$
Strong ergodicity L^1-exp. conv.	$(*)$ & $\int_0^\infty \mu[x,\infty)e^{-C(x)} < \infty$
Nash inequality	$(*)$ & $\sup_{x>0} \mu[x,\infty)^{(\nu-2)/\nu} \int_0^x e^{-C} < \infty(\varepsilon)$

$(*)$ &. $\quad (\varepsilon)$.

$$\text{Interval: } [0, D], \quad D \leqslant \infty.$$

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad a(x) > 0.$$

Eigenvalue: $Lf = -\lambda_0 f$,

Dirichlet: $f(0) = 0$; Neumann: $f'(D) = 0$.

Poincaré inequality:

$$\int_0^D f^2 \mathrm{d}\mu = \int_0^D f^2 e^C / a \leqslant A \int_0^D f'^2 e^C =: AD(f)$$

$$f(0) = 0, \; f \in \mathbb{C}_{\mathrm{d}}[0, D].$$

$\mathbb{C}_{\mathrm{d}}[0, D]$: Continuous, a.e. diff. compact supp.

Logarithmic Sobolev inequality:

$$\int_0^D f^2 \log \frac{f^2}{\pi(f^2)} \mathrm{d}\mu \leqslant A_{LS} \int_0^D f'^2 e^C =: A_{LS} D(f)$$

F-Sobolev inequality:

$$\int_0^D f^2 F\big(f^2/\pi(f^2)\big) \mathrm{d}\mu \leqslant A_F D(f)$$

$f(0) = 0$, $f \in \mathbb{C}_{\mathrm{d}}[0, D]$. $\pi(\mathrm{d}x) = \mu(\mathrm{d}x)/Z$.

Importance:

Poincaré inequality $\iff \|P_t f\|_2 \leqslant \|f\|_2 e^{-\varepsilon t}$.

$\|\cdot\|_p$ w.r.t. μ. For diffusions,

Log.Sobolev $\iff \mathrm{Ent}(P_t f) \leqslant \mathrm{Ent}(f) e^{-\varepsilon t}$.

F-Sobolev inequality $\iff$ Discrete spectrum

Examples: Diffusions on $[0, \infty)$

	Erg.	Exp.erg. Poinc.	LogS Ent.	Str.erg. L^1Exp.	Nash
$b(x) = 0$ $a(x) = x^\gamma$	$\gamma > 1$	$\gamma \geqslant 2$	$\gamma > 2$	$\gamma > 2$	$\gamma > 2$
$b(x) = 0$ $a = x^2 \log^\gamma x$	$\surd$	$\gamma \geqslant 0$	$\gamma \geqslant 1$	$\gamma > 1$	$\times$
$a(x) = 1$ $b(x) = -b$	$\surd$	$\surd$	$\times$	$\times$	$\times$

2. Extension. Banach spaces.

Poincaré: $\int_0^D f^2 \mathrm{d}\mu = \int_0^D f^2 e^C/a \leqslant AD(f)$

LogS.: $\int_0^D f^2 \log\big(f^2/\pi(f^2)\big)\mathrm{d}\mu \leqslant A_{LS}D(f)$

F-Sobolev: $\int_0^D f^2 F\big(f^2/\pi(f^2)\big)\mathrm{d}\mu \leqslant A_F D(f)$

Poincaré inequality: $L^2(\mu)$

Log-Sobolev: $\left\{f : \int_0^D f^2 \log(1+f^2)\mathrm{d}\mu < \infty\right\}$

F-Sobolev ineq.: $\left\{f : \int_0^D f^2 F(f^2)\mathrm{d}\mu < \infty\right\}$

Nash inequality: $L^p(\mu)$

$$\|f^2\|_{\mathbb{B}} \leqslant A_{\mathbb{B}} D(f).$$

Banach space: $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$, $[0, D] \to \mathbb{R}$.

- $1 \in \mathbb{B}$.
- $h \in \mathbb{B}$, $|f| \leqslant |h| \Longrightarrow f \in \mathbb{B}$.
- $\|f\|_{\mathbb{B}} = \sup_{g \in \mathscr{G}} \int_0^D |f| g \mathbf{d}\mu$,
  $\mathscr{G} \subset \{g : [0, D] \to \mathbb{R}_+\}$:
- $\mathscr{G} \ni g_0$ **with** $\inf g_0 > 0$.

Example: $\mathbb{B} = L^p(\mu)$, $\mathscr{G} =$ **unit ball in** $L_+^q(\mu)$.

Poincaré-type inequality in Banach spaces:

$$\|f^2\|_{\mathbb{B}} \leqslant A_{\mathbb{B}} D(f),$$

$$f(0) = 0, \ f \in \mathbb{C}_{\mathrm{d}}[0, D].$$

$\mathbb{C}_{\mathrm{d}}[0, D]$: Continuous, a.e.-diff., compact supp.

Variational formula for lower bounds:

$$A_{\mathbb{B}} = \sup \left\{ \frac{\|f^2\|_{\mathbb{B}}}{D(f)} : f \in \mathbb{C}_{\mathrm{d}}[0, D], \right.$$
$$\left. f(0) = 0, \ 0 < D(f) < \infty \right\}.$$

Upper bounds are more useful but much harder to handle.

Theorem. (1) Variational formula for upper bounds

$$A_{\mathbb{B}} \leqslant \inf_{f\in\mathscr{F}} \sup_{x\in(0,D)} \frac{e^{-C(x)}}{f'(x)} \big\| f I_{(x,D)} \big\|_{\mathbb{B}},\ C(x) = \int_0^x \frac{b}{a}$$

$$\mathscr{F} = \big\{ f \in C[0,D] : f(0) = 0, f'|_{(0,D)} > 0 \big\}.$$

(2) Explicit bounds.

$$B_{\mathbb{B}} \leqslant C_{\mathbb{B}} \leqslant A_{\mathbb{B}} \leqslant D_{\mathbb{B}} \leqslant 4B_{\mathbb{B}}.$$

$$B_{\mathbb{B}} = \sup_{x\in(0,D)} \varphi(x) \big\| I_{(x,D)} \big\|_{\mathbb{B}},\ \ \varphi(x) = \int_0^x e^{-C},$$

$$C_{\mathbb{B}} = \sup_{x\in(0,D)} \varphi(x)^{-1} \big\| \varphi(x\wedge\cdot)^2 \big\|_{\mathbb{B}},$$

$$D_{\mathbb{B}} = \sup_{x\in(0,D)} \varphi(x)^{-1/2} \big\| \sqrt{\varphi}\, \varphi(x\wedge\cdot)^2 \big\|_{\mathbb{B}}.$$

(3) Criterion. $A_{\mathbb{B}} < \infty$ iff $B_{\mathbb{B}} < \infty$.

Back to Bobkov & Götze, JFA 1999.

Theorem. (1) Second variational formula for upper bounds.

$$A_{\mathbb{B}} \leqslant \inf_{f\in\mathscr{F}'} \sup_{x\in(0,D)} f(x)^{-1}\big\|f\varphi(x\wedge\cdot)\big\|_{\mathbb{B}},$$

$$\mathscr{F}' = \big\{f \in C[0,D], f(0)=0, f\big|_{(0,D)} > 0\big\} \supset \mathscr{F}.$$

(2) Approximating procedure.

Let $B_{\mathbb{B}} < \infty$. Define $f_0 = \sqrt{\varphi}$,

$f_n(x) = \|f_{n-1}\varphi(x\wedge\cdot)\|_{\mathbb{B}}$,

$D_{\mathbb{B}}(n) = \sup_{x\in(0,D)} f_n/f_{n-1}$ for $n \geqslant 1$. Then,

$$4B_{\mathbb{B}} \geqslant D_{\mathbb{B}}(n) \downarrow \lim_{n\to\infty} D_{\mathbb{B}}(n) \geqslant A_{\mathbb{B}}.$$

Proof. For ordinary Poincaré inequality:

$$\int_0^D f^2 \mathrm{d}\mu = \int_0^D f^2 e^C / a \leqslant A \int_0^D f'^2 e^C, \;\; f(0) = 0.$$

$$A \leqslant \inf_{f \in \mathscr{F}} \sup_{x \in (0,D)} \frac{e^{-C(x)}}{f'(x)} \int_x^D f e^C / a.$$

Fix $g > 0$. Introduce a transform:

$$b \to b/g, \qquad a \to a/g > 0. \quad \text{Then}$$

$$C(x) \longrightarrow C_g(x) = \int_0^x \frac{b/g}{a/g} = C(x),$$

$$\int_0^D f^2 e^C / a \longrightarrow \int_0^D f^2 g e^C / a = \int_0^D f^2 g \mathrm{d}\mu,$$

$$A \longrightarrow A_g \leqslant \inf_{f \in \mathscr{F}} \sup_{x \in (0,D)} \frac{e^{-C(x)}}{f'(x)} \int_x^D f g \mathrm{d}\mu$$

Make supremum with respect to $g \in \mathscr{G}$.

$\sup_{g\in\mathscr{G}} \int_0^D f^2 g \mathbf{d}\mu = \|f^2\|_{\mathbb{B}}$.

$$A_{\mathbb{B}} = \sup_g A_g \leqslant \sup_g \inf_f \sup_x \leqslant \inf_f \sup_g \sup_x$$

$$= \inf_f \sup_x \frac{e^{-C(x)}}{f'(x)} \sup_g \int_0^D f I_{(x,D)} g \mathbf{d}\mu.$$

$$= \inf_f \sup_x \frac{e^{-C(x)}}{f'(x)} \|f I_{(x,D)}\|_{\mathbb{B}}.$$

Lucky point! Dual for lower bounds failed!

$0 \leqslant g \to g + 1/n, \qquad n \geqslant 1.$

3. Neumann Case

$f(0) = 0 \to f'(0) = 0$. $[0, D] \to \mathbb{R}$. $\lambda_0 = 0$.

First non-trivial eigenvalue:

$Lf = -\lambda_1 f$, $f \neq$constant. Spectral gap.

Poincaré inequality:

$$\int_{\mathbb{R}} \big(f - \pi(f)\big)^2 \mathrm{d}\mu \leqslant \overline{A} D(f), \quad f \in \mathbb{C}_d(\mathbb{R})$$

$$\overline{A} \leqslant \inf_{f \in \mathscr{F}''} \sup_{x \in (0,D)} \frac{e^{-C(x)}}{f'(x)} \int_x^D f e^C / a.$$

Banach form:

$$\big\| \big(f - \pi(f)\big)^2 \big|_{\mathbb{B}} \leqslant \overline{A}_{\mathbb{B}} D(f), \quad f \in \mathbb{C}_d(\mathbb{R}).$$

Splitting. Let $\theta \in \mathbb{R}$, $f(\theta) = 0$. $(-\infty, \theta)$, (θ, ∞).

Neumann→Dirichlet. $A_{\mathbb{B}}^{k\theta}$, $B_{\mathbb{B}}^{k\theta}$, $C_{\mathbb{B}}^{k\theta}$, $D_{\mathbb{B}}^{k\theta}$.

$$B_{\mathbb{B}}^{k\theta} \leqslant C_{\mathbb{B}}^{k\theta} \leqslant A_{\mathbb{B}}^{k\theta} \leqslant D_{\mathbb{B}}^{k\theta} \leqslant 4B_{\mathbb{B}}^{k\theta},\ k = 1, 2.$$

Theorem. $\overline{A}_{\mathbb{B}} < \infty \iff B_{\mathbb{B}}^{1\theta} \vee B_{\mathbb{B}}^{2\theta} < \infty$.

$$\overline{A}_{\mathbb{B}} \leqslant A_{\mathbb{B}}^{1\theta} \vee A_{\mathbb{B}}^{2\theta},$$
$$\overline{A}_{\mathbb{B}} \geqslant \max\left\{\tfrac{1}{2}\big(A_{\mathbb{B}}^{1\theta} \wedge A_{\mathbb{B}}^{2\theta}\big),\ K_{\theta}\big(A_{\mathbb{B}}^{1\theta} \vee A_{\mathbb{B}}^{2\theta}\big)\right\}.$$

Birth-death processes

$$i \to i+1 \quad \text{at rate} \quad b_i = q_{i,i+1} > 0$$
$$\to i-1 \quad \text{at rate} \quad a_i = q_{i,i-1} > 0.$$

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n}, \quad \pi_n = \frac{\mu_n}{Z}, \; n \geqslant 1.$$

Theorem. State space: $\{0, 1, 2, \ldots\}$. Let c_1, c_2:
$|\pi(f)| \leqslant c_1 \|f\|_{\mathbb{B}}$, $|\pi(f I_{E_1})| \leqslant c_2 \|f I_{E_1}\|_{\mathbb{B}}$, $\forall f \in \mathbb{B}$, $E_1 = \{1, 2, \cdots\}$. Then, we have

$$\overline{A}_{\mathbb{B}} \leqslant \left(1 + \sqrt{c_1 \|1\|_{\mathbb{B}}}\,\right)^2 A_{\mathbb{B}}, \qquad (f_0 = 0)$$
$$\overline{A}_{\mathbb{B}} \geqslant \max\left\{\|1\|_{\mathbb{B}}^{-1}, \left(1 - \sqrt{c_2(1-\pi_0)\|1\|_{\mathbb{B}}}\,\right)^2\right\} A_{\mathbb{B}}.$$

In particular, $\overline{A}_{\mathbb{B}} < \infty$ iff $B_{\mathbb{B}} < \infty$.

4. Orlicz spaces.

Orlicz spaces$\subset$Banach spaces. Interpolation of $L^p(\mu)$-spaces.

(Nice Young)N-function Φ: 7 conditions:

- Non-negative, continuous, convex and even
- $\Phi(x) = 0$ iff $x = 0$
- $\lim_{x\to 0} \Phi(x)/x = 0$
- $\lim_{x\to\infty} \Phi(x)/x = \infty$

Complementary N-function:

$$\Phi_c(y) = \sup\{x|y| - \Phi(x) : x \geqslant 0\}, \quad y \in \mathbb{R}.$$

Growth (Δ_2)-condition: $\displaystyle\sup_{x\gg 1} \left[\frac{\Phi(2x)}{\Phi(x)} \text{ or } \right] \frac{x\Phi'(x)}{\Phi(x)} < \infty.$

Examples

- $\Phi(x) = |x|^p/p$, $\Phi_c(y) = |y|^q/q$, $\quad \frac{1}{p} + \frac{1}{q} = 1$
- $\Phi(x) = (1+|x|)\log(1+|x|) - |x|$,
  $\Phi_c(y) = e^{|y|} - |y| - 1$
- $\Phi(x) = |x|\log(1+|x|)$, $\Phi_c = ?$
- $\Phi(x) = x^2\log(1+x^2)$, $\Phi_c = ?$

Orlicz spaces

$$(E,\mu). \quad L^\Phi(\mu) = \left\{ f : \int_E \Phi(f)\mathrm{d}\mu < \infty \right\}$$

$$\|f\|_\Phi = \sup_{g\in\mathscr{G}} \int_E |f| g \mathrm{d}\mu$$

$$\mathscr{G} = \text{unit ball in } L_+^{\Phi_c}(\mu) = \left\{ g \geqslant 0 : \int_E \Phi_c(g)\mathrm{d}\mu \leqslant 1 \right\}.$$

5. Nash inequality

Theorem (Varopoulos, N. (80's); Bakry, D., Coulhon, T., Ledoux, M. and Saloff-Coste, L. (1995)).

When $\nu > 2$, Nash inequality:

$$\|f - \pi(f)\|^{2+4/\nu} \leqslant C_1 D(f) \|f\|_1^{4/\nu}$$

is equivalent to the Sobolev-type inequality:

$$\|f - \pi(f)\|^2_{\nu/(\nu-2)} \leqslant C_2 D(f),$$

where $\|\cdot\|_p$ is the $L^p(\mu)$-norm. $\quad p = \nu/(\nu - 2)$.

Orlicz space $L^\Phi(\mu)$ with $\Phi(x) = |x|^p/p$:

$$\big\|(f - \pi(f))^2\big\|_\Phi \leqslant \overline{A}_\nu D(f).$$

Theorem (Mao, Y. H. (2002), C. (2002)). Let $\nu > 2$.
(1) Nash inequality on $\mathbb{R}$ holds iff $B_\nu^{1\theta} \vee B_\nu^{2\theta} < \infty$.
(2) $\frac{1}{2}(B_\nu^{1\theta} \wedge B_\nu^{2\theta}) \leqslant \overline{A}_\nu \leqslant 4(B_\nu^{1\theta} \vee B_\nu^{2\theta})$.
(3) $\overline{A}_\nu \geqslant \left[1 - \left(\frac{Z_{1\theta} \vee Z_{2\theta}}{Z}\right)^{1/2+1/\nu}\right]^2 (B_\nu^{1\theta} \vee B_\nu^{2\theta})$.
In particular, if θ is the medium of μ, then

$$\overline{A}_\nu \geqslant \left[1 - (1/2)^{1/2+1/\nu}\right]^2 (B_\nu^{1\theta} \vee B_\nu^{2\theta}),$$

$$B_\nu^{1\theta} = \sup_{\theta < x} \varphi^{1\theta}(x) \mu(\theta, x)^{(\nu-2)/\nu},$$

$$C(x) = \int_\theta^x b/a, \quad \varphi^{1\theta}(x) = \int_\theta^x e^{-C}.$$

$$Z_{1\theta} = \int_\theta^\infty e^C/a, \ Z = Z_{1\theta} + Z_{2\theta}.$$

6. Logarithmic Sobolev inequality.

Lemma (Rothaus, O. S. (1985), Bobkov, S. G. & Götze, F. (1999), C. (2002)).

$$\frac{2}{5}\|(f-\pi(f))^2\|_\Phi \leqslant \mathscr{L}(f) \leqslant \frac{51}{20}\|(f-\pi(f))^2\|_\Phi,$$

where

$$\Phi(x) = |x|\log(1+|x|),$$

$$\mathscr{L}(f) = \sup_{c\in\mathbb{R}} \mathbf{Ent}\big((f+c)^2\big),$$

$$\mathbf{Ent}(f) = \int_{\mathbb{R}} f\log\frac{f}{\|f\|_1}\mathrm{d}\mu, \quad f \geqslant 0.$$

Theorem (Bobkov, S. G. & Götze, F. (1999), Mao, Y. H. (2002), C. 2002). Log-Sobolev on $\mathbb{R}$ holds iff

$$\sup_{x\in(\theta,\infty)} \mu(x,\infty)\log\frac{1}{\mu(x,\infty)}\int_\theta^x e^{-C}<\infty \ \&$$

$$\sup_{x\in(-\infty,\theta)} \mu(-\infty,x)\log\frac{1}{\mu(-\infty,x)}\int_x^\theta e^{-C}<\infty.$$

- Let $\bar\theta$: $B_{\mathbb{B}}^{1\bar\theta}=B_{\mathbb{B}}^{2\bar\theta}$. Then $\frac{1}{5}B_{\mathbb{B}}^{1\bar\theta}\leqslant A_{LS}\leqslant\frac{51}{5}B_{\mathbb{B}}^{1\bar\theta}$.
- Let $\theta=0$ be the medium of μ. Then

$$\frac{(\sqrt{2}-1)^2}{5}\big(B_{\mathbb{B}}^{1\theta}\vee B_{\mathbb{B}}^{2\theta}\big)\leqslant A_{LS}\leqslant\frac{51}{5}\big(B_{\mathbb{B}}^{1\theta}\vee B_{\mathbb{B}}^{2\theta}\big).$$

$\mathbb{B}=L^{\Phi}(\mu)$. **Ratio** ≈ 300.

7. Proof of a key formula

Key formula (Probabilistic(C. & F. Y. Wang, 1997), analytic(C. 1999)).

$$\overline{A} \leqslant \inf_{f \in \mathscr{F}''} \sup_{x \in (0,D)} I(f)(x)$$

$$\mathscr{F}'' = \{f \in C[0,D] : \pi(f) \geqslant 0, f'|_{(0,D)} > 0\},$$

$$I(f)(x) = \frac{e^{-C(x)}}{f'(x)} \int_x^D \frac{f e^C}{a}.$$

Proof. Let $g \in C[0,D] \cap C^1(0,D)$, $\pi(g) = 0$ and $\pi(g^2) = 1$. For every $f \in \mathscr{F}''$,

$$
\begin{aligned}
1 &= \frac{1}{2}\int_0^D \pi(\mathrm{d}x)\pi(\mathrm{d}y)[g(y)-g(x)]^2 \\
&= \int_{\{x\leqslant y\}} \pi(\mathrm{d}x)\pi(\mathrm{d}y)\left(\int_x^y \frac{g'(u)\sqrt{f'(u)}}{\sqrt{f'(u)}}\mathrm{d}u\right)^2 \\
&\leqslant \int_{\{x\leqslant y\}} \pi(\mathrm{d}x)\pi(\mathrm{d}y)\int_x^y \frac{g'(u)^2}{f'(u)}\mathrm{d}u \int_x^y f'(\xi)\mathrm{d}\xi
\end{aligned}
$$

(by Cauchy-Schwarz inequality)

$$
\begin{aligned}
&= \int_{\{x\leqslant y\}} \pi(\mathrm{d}x)\pi(\mathrm{d}y)\int_x^y g'(u)^2 e^{C(u)}\frac{e^{-C(u)}}{f'(u)}\mathrm{d}u \\
&\quad \times [f(y)-f(x)]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^D a(u) g'(u)^2 \pi(\mathrm{d}u) \frac{Ze^{-C(u)}}{f'(u)} \\
&\quad \times \int_0^u \pi(\mathrm{d}x) \int_u^D \pi(\mathrm{d}y) \left[f(y) - f(x) \right] \\
&\leqslant D(g) \sup_{u \in (0,D)} \frac{Ze^{-C(u)}}{f'(u)} \\
&\quad \times \int_0^u \pi(\mathrm{d}x) \int_u^D \pi(\mathrm{d}y) \left[f(y) - f(x) \right] \\
&\leqslant D(g) \sup_{x \in (0,D)} I(f)(x) \quad \big(\text{since } \pi(f) \geqslant 0\big).
\end{aligned}
$$

$$D(g)^{-1} \leqslant \sup_{x\in(0,D)} I(\bar{f})(x)$$

$$\overline{A} = \sup_{g:\ \pi(g)=0,\ \pi(g^2)=1} D(g)^{-1} \leqslant \sup_{x\in(0,D)} I(\bar{f})(x)$$

$$\overline{A} \leqslant \inf_{f\in\mathscr{F}''} \sup_{x\in(0,D)} I(\bar{f})(x).$$

Done! Equality holds for continuous a and b.

How to find out the formula?

Higher dimension. Coupling methods.

Markov process with operator L. Two steps:

- Choose a good coupling of L, say $\widetilde{L}$.
- Choose a good distance ρ.

$$\widetilde{L}\rho \leqslant -\alpha\rho \Longrightarrow \lambda_1 \geqslant \alpha.$$

Dealing with one-dim. process valued on $[0, D]$.
Exponential convergence rate $= \lambda_0$.
Provide a solution to the second problem.
ICM: Preprint Buffet.

The Convergence Stronger or Weaker than Exp One

(The inequalities stronger or weaker than Poincaré's One)

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Contents

Part I. Stronger convergence. From Hilbert space to Banach (Orlicz) spaces

1. Statement of the results

Theorem [V. G. Maz'ya, 1973; Z. Vondraček, 1996; M. Fukushima and T. Uemura 2002]. $(E, \mathscr{E}, \mu)$. For regular transient Dirichlet form $(D, \mathscr{D}(D))$, the optimal constant A in the Poincaré inequality

$$\|f\|^2 = \int_E f^2 \mathrm{d}\mu \leqslant A D(f), \quad A = \lambda_0{}^{-1}$$

satisfies $B \leqslant A \leqslant 4B$, where $B = \sup_{\text{compact } K} \dfrac{\mu(K)}{\mathrm{Cap}(K)}$.

Dim one: G.H.Hardy,1920; B.Muckenhoupt,1970.

Extension to Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$:

$$\|f^2\|_{\mathbb{B}} \leqslant A_{\mathbb{B}} D(f).$$

Examples:

- Nash- or Sobolev-type ineq. $\mathbb{B} = L^p(\mu), p \geqslant 1.$
- Log Sobolev inequality: $\displaystyle\int_E f^2 \log f^2 \mathrm{d}\mu.$
- F-Sobolev inequality: $\displaystyle\int_E f^2 F\big(f^2\big) \mathrm{d}\mu.$

$L^{1+\varepsilon} \subset$ LHS of LogS $\subset L^1 \longrightarrow$ Orlicz space.

Interpolation. Stronger inequality (convergence).

Assumptions on $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$:

$$
\begin{aligned}
&(1) \quad 1 \in \mathbb{B};\\
&(2) \quad \text{If } h \in \mathbb{B} \text{ and } |f| \leqslant h, \ \text{then } f \in \mathbb{B};\\
&(3) \quad \|f\|_{\mathbb{B}} = \sup_{g \in \mathscr{G}} \int_E |f| g \, \mathrm{d}\mu \quad \text{for fixed } \mathscr{G}.
\end{aligned}
$$

Theorem [C., in preparation]. For regular transient Dirichlet form $(D, \mathscr{D}(D))$, the optimal constant $A_{\mathbb{B}}$ in: $\|f^2\|_{\mathbb{B}} \leqslant A_{\mathbb{B}} D(f)$, satisfies

$$B_{\mathbb{B}} \leqslant A_{\mathbb{B}} \leqslant 4 B_{\mathbb{B}}$$

$$B_{\mathbb{B}} := \sup_{\text{compact } K} \frac{\|I_K\|_{\mathbb{B}}}{\text{Cap}(K)}.$$

Math tool: Harmonic analysis. Potential theory.

Neumann (or closed) case. Let $E_1 \subset E$, $E_2 = E_1^c$.
Restriction of $\mathbb{B}$ on E_i: $(\mathbb{B}^i, \|\cdot\|_{\mathbb{B}^i}, \mu^i)$, $i = 1, 2$.
Corresponding Dirichlet form on E_i, $A_{\mathbb{B}^i}$, $B_{\mathbb{B}^i}$.
Let $\mu(E) < \infty$, set $\pi = \mu/\mu(E), \bar{f} = f - \pi(f)$.
Define

$$c_1 : |\pi(f)| \leqslant c_1 \|f\|_{\mathbb{B}}, \qquad f \in \mathbb{B}.$$
$$c_2(G) : |\pi(f I_G)| \leqslant c_2(G) \|f I_G\|_{\mathbb{B}}, \ f \in \mathbb{B}, \ G \subset E.$$

Theorem [C.]. $(D, \mathscr{D}(D))$: a regular Dirichlet form.
Assume $c_2(E_i)\pi(E_i)\|1\|_{\mathbb{B}} < 1, \forall i$. $\|\bar{f}\|_{\mathbb{B}} \leqslant \overline{A}_{\mathbb{B}} D(f)$.

$$\overline{A}_{\mathbb{B}} \geqslant (K_1 \wedge K_2)(A_{\mathbb{B}^1} \vee A_{\mathbb{B}^2}) \geqslant (\cdots)(B_{\mathbb{B}^1} \vee B_{\mathbb{B}^2}),$$
$$\overline{A}_{\mathbb{B}} \leqslant 4\left(1 + \sqrt{c_1 \|1\|_{\mathbb{B}}}\right)^2 (A_{\mathbb{B}^1} \vee A_{\mathbb{B}^2}),$$
$$K_i = \left(1 - \sqrt{c_2(E_i)\pi(E_i)\|1\|_{\mathbb{B}}}\right)^2, \quad i = 1, 2.$$

LogS: $\Phi(x) = |x|\log(1+|x|)$. Complementary: Φ_c.
Orlicz space $\mathbb{B} = L^{\Phi}(\mu) = \{f : \int \Phi(f)\mathrm{d}\mu < \infty\}$,
$\|f\|_{\mathbb{B}} = \|f\|_{\Phi} = \sup\left\{\int |f|g\mathrm{d}\mu : \int \Phi_c(g)\mathrm{d}\mu \leqslant 1\right\}$.

Theorem [C.]. The optimal A_{Log} in LogS satisfies

$$A_{\mathrm{Log}} \geqslant \frac{(\sqrt{2}-1)^2}{5}\left(B_\Phi^1 \vee B_\Phi^2\right), \quad \mu(E_i) \in (0, 1/2]$$

$$A_{\mathrm{Log}} \leqslant \frac{204}{5}\Big(1 + \Phi^{-1}\big(\mu(E)^{-1}\big)\Big)^2\left(B_\Phi^1 \vee B_\Phi^2\right),$$

$$B_\Phi^i = \sup_{\text{compact } K \subset E_i} \frac{M(\mu(K))}{\mathrm{Cap}(K)},$$

$$M(x) = \frac{1}{2}\left(\sqrt{1+4x}-1\right) + x\log\left(1+\frac{1+\sqrt{1+4x}}{2x}\right)$$

$$\sim x\log x^{-1}, \quad \text{as } x \to 0.$$

2. Sketch of the proofs:

$$\|f\|^2 = \int_E f^2 \mathrm{d}\mu \leqslant AD(f).$$

Replace μ by $\mu_g = g\mu$. Then

$$\int_E f^2 \mathrm{d}\mu_g \leqslant A_g D(f).$$

$$\text{LHS}: \quad \sup_{g\in\mathscr{G}} \int_E f^2 \mathrm{d}\mu_g$$

$$= \sup_{g\in\mathscr{G}} \int_E f^2 g \mathrm{d}\mu = \|f^2\|_{\mathbb{B}}.$$

Next,

$$A_{\mathbb{B}} = \sup_{g\in\mathscr{G}} A_g \geqslant \sup_g B_g$$
$$= \sup_g \sup_K \frac{\mu_g(K)}{\mathrm{Cap}(K)} = \sup_K \sup_g$$
$$= \sup_K \frac{\|I_K\|_{\mathbb{B}}}{\mathrm{Cap}(K)} = B_{\mathbb{B}}.$$

$$\mathrm{Cap}(K) = \inf\left\{D(f) : f \in \mathscr{F} \cap C_0(E), f|_K \geqslant 1\right\}.$$

Similarly, $\|A\|_{\mathbb{B}} \leqslant 4B_{\mathbb{B}}$.

There is still a gap in the “proof”. Use different one.

Neumann case.

Proposition. $(E, \mathscr{E}, \pi)$ prob sp, $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ Banach.

- Let c_1: $|\pi(f)| \leqslant c_1\|f\|_{\mathbb{B}}$, $\forall f \in \mathbb{B}$. Then

$$\|\bar{f}^2\|_{\mathbb{B}} \leqslant \left(1 + \sqrt{c_1\|1\|_{\mathbb{B}}}\,\right)^2 \|f^2\|_{\mathbb{B}}.$$

- Let $c_2(G)$: $|\pi(f I_G)| \leqslant c_2(G)\|f I_G\|_{\mathbb{B}}$, $\forall f \in \mathbb{B}$. If $c_2(G)\pi(G)\|1\|_{\mathbb{B}} < 1$, then $\forall f$: $f|_{G^c} = 0$,

$$\|f^2\|_{\mathbb{B}} \leqslant \|\bar{f}^2\|_{\mathbb{B}} / \left[1 - \sqrt{c_2(G)\pi(G)\,\|1\|_{\mathbb{B}}}\,\right]^2.$$

Proof. $\pi(f)^2 \leqslant \pi\big(f^2\big) \leqslant c_1\|f^2\|_{\mathbb{B}}$. $\forall p, q > 1$ with $(p-1)(q-1) = 1$, $(x+y)^2 \leqslant px^2 + qy^2$,

$$\|\bar{f}^2\|_{\mathbb{B}} \leqslant p\|f^2\|_{\mathbb{B}} + q\pi(f)^2\|1\|_{\mathbb{B}} \leqslant \big(p + c_1 q\|1\|_{\mathbb{B}}\big)\|f^2\|_{\mathbb{B}}.$$

Minimizing RHS w.r.t. p and q, get first assertion.

3. Comparison with Cheeger's method

For F-Sobolev, if $2F' + xF'' \geqslant 0$ on $[0, \infty)$, $\lim_{x\to 0} F(x) = 0$, $\lim_{x\to\infty} F(x) = \infty$, $\sup_{x\gg 1} xF'(x)/F(x) < \infty$, take $\Phi(x) = |x|F(|x|)$. Then isoperimetric constant is given by

$$B_\Phi = \sup_{\text{compact } K\subset E_i} \frac{\alpha_*(K)^{-1} + \mu(K)F(\alpha_*(K))}{\text{Cap}(K)},$$

where $\alpha_*(K)$ is the minimal positive root of $\alpha^2 F'(\alpha) = \mu(K)$.

More precise but less explicit than Cheeger's mtd. Price to pay for the higher-dimensional situation. Need more geometry. Correct iso constant.

$$D^{(\alpha)}(f) = \frac{1}{2}\int J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2 + \int K^{(\alpha)}(\mathrm{d}x) f(x)^2$$

$$[J^{(1)}(\mathrm{d}x, E) + K^{(1)}(\mathrm{d}x)]/\pi(\mathrm{d}x) \leqslant 1,$$

$$\lambda_0^{(\alpha)} = \inf\left\{D^{(\alpha)}(f) : \|f\| = 1\right\}. \; c_1 = \sup_{x \geqslant 0} \frac{x F'_{\pm}(x)}{F(x)} < \infty$$

Theorem [C., 2000: Log Sobolev, Theorem 2.1].
The optimal A_F in $\int f^2 F\big(f^2\big)\mathrm{d}\pi \leqslant A_F D(f)$:

$$A_F \geqslant \sup_{\pi(G)>0} \frac{\pi(G) F\big(\pi(G)^{-1}\big)}{J(G \times G^c) + K(G)},$$

$$A_F \leqslant \sup_{\pi(G)>0} \frac{4(1+c_1^2)\big(2-\lambda_0^{(1)}\big)\pi(G)^2 F\big(\pi(G)^{-1}\big)}{\big[J^{(1/2)}(G \times G^c) + K^{(1/2)}(G)\big]^2}.$$

Comments on F-Sobolev inequalities

- [F. Y. Wang, 2000; — and F. Z. Gong, 2002]. $\sigma_{\text{ess}} = \emptyset \implies F$-Sobolev. Inverse statement holds once having density w.r.t. π.
- [F. Y. Wang, 1999]. Estimating λ_j, $j \geqslant 1$.
- [F. Y. Wang, 2002]. Beckner-type inequality. Sometimes additive. Infinite dimension.
- Interpolation between LogS and Poincaré, meaningful for non-local operators to study the exponential convergence in entropy.

4. One- or infinite-dimensional cases

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \theta \in (p,q),$$

$$C(x) = \int_\theta^x \frac{b}{a}; \qquad \mu(m,n) = \int_m^n e^C/a;$$

$$B_\Phi^{1\theta} = \sup_{x\in(\theta,q)} M(\mu(\theta,x)) \int_\theta^x e^{-C}, \qquad \text{explicit!}$$

$$B_\Phi^{2\theta} = \sup_{x\in(p,\theta)} M(\mu(x,\theta)) \int_x^\theta e^{-C}.$$

$$M(x) = \frac{1}{2}\left(\sqrt{1+4x}-1\right) + x\log\left(1+\frac{1+\sqrt{1+4x}}{2x}\right).$$

Theorem [S. G. Bobkov & F. Götze, 1999; Y. H. Mao, 2002; C., 2002]. Ergodic.

Assume that $\theta = 0$ is the medium of μ. Then,

$$\frac{(\sqrt{2}-1)^2}{5}\big(B_\Phi^{1\theta} \vee B_\Phi^{2\theta}\big) \leqslant A_{\mathsf{Log}} \leqslant \frac{51}{5}\big(B_\Phi^{1\theta} \vee B_\Phi^{2\theta}\big).$$

Let $\bar\theta$ be the root of $B_\Phi^{1\theta} = B_\Phi^{2\theta}$, $\theta \in [p, q]$. Then,

$$\frac{1}{5}B_\Phi^{1\bar\theta} \leqslant A_{\mathsf{Log}} \leqslant \frac{51}{5}B_\Phi^{1\bar\theta}.$$

Eleven criteria for one-dimensional diffusions

Property	Criterion
Uniqueness	$\int_0^\infty \mu[0,x]e^{-C(x)} = \infty \quad (*)$
Recurrence	$\int_0^\infty e^{-C(x)} = \infty$
Ergodicity	$(*)\ \&\ \mu[0,\infty) < \infty$
Exp. ergodicity Poincaré inequality	$(*)\ \&\ \sup_{x>0}\mu[x,\infty)\int_0^x e^{-C} < \infty$
Discrete spectrum	$(*)\ \&\ \lim_{n\to\infty}\sup_{x>n}\mu[x,\infty)\int_n^x e^{-C} = 0$
LogS inequality Exp. conv. in entropy	$(*)\ \&\ \sup_{x>0}\mu[x,\infty)\log[\mu[x,\infty)^{-1}]\int_0^x e^{-C} < \infty$
Strong ergodicity L^1-exp. conv.	$(*)\ \&\ \int_0^\infty \mu[x,\infty)e^{-C(x)} < \infty$
Nash inequality	$(*)\ \&\ \sup_{x>0}\mu[x,\infty)^{(\nu-2)/\nu}\int_0^x e^{-C} < \infty(\varepsilon)$

$(*)$ &. $\quad (\varepsilon)$.

Theorem [C., 1996, 2000, 2001]. $[0,\infty),\ f(0)=0$

- **Dual variational formulas.**
  $\lambda_0 \geqslant \sup_{f\in\mathscr{F}} \inf_{x>0} e^{C(x)} f'(x)\big/\int_x^\infty [fe^C/a]$. $\quad \lambda_0 = A^{-1}$

  $\lambda_0 \leqslant \inf_{f\in\mathscr{F}'} \sup_{x>0} e^{C(x)} f'(x)\big/\int_x^\infty [fe^C/a]$.

  Equalities hold for continuous a and b.
- **Explicit estimates.** $\delta^{-1} \geqslant \lambda_0 \geqslant (4\delta)^{-1}$, where $\delta = \sup_{x>0} \int_0^x e^{-C} \int_x^\infty e^C/a$.
- **Approximation procedure.** $\exists$ explicit η_n', η_n'' such that $\quad \delta^{-1} \geqslant {\eta_n'}^{-1} \geqslant \lambda_0 \geqslant {\eta_n''}^{-1} \geqslant (4\delta)^{-1}$.

Apply to infinite dimensional model

Spin potential: $u(x) = x^4 - \beta x^2, \quad x, \beta \in \mathbb{R}$.
Hamiltonian: $-2J \sum_{\langle ij \rangle} x_i x_j, \ \ J \geqslant 0$.
Theorem [C., in preparation].

$$\lambda_1(\text{of the conditional Gibbsian}) \\ \geqslant \frac{\sqrt{\beta^2+8}-\beta}{\sqrt{e}} \exp\left[-\frac{1}{8}\beta\left(\beta+\sqrt{\beta^2+8}\right)\right] \\ - 4dJ$$

uniformly in boxes and arbitrary boundary.

Part II. Weaker (slower) convergence

1. General convergence speed

Let $\xi(t) \downarrow 0$ as $t \uparrow \infty$. Consider the decay

$$\|P_t \bar{f}\|^2 = \|P_t f - \pi(f)\|^2 \leqslant C V(f) \xi(t).$$

What is analog of Poincaré inequality?

$$V(f) := \sup_{t>0} \xi(t)^{-1} \|P_t \bar{f}\|^2$$

$\Longrightarrow V(\alpha f + \beta) = \alpha^2 V(f)$ (Hom of degree 2)

$$V(f) := \|\bar{f}\|^2 \Longrightarrow \xi(t) \sim e^{-\varepsilon t}.$$

Because $0 \leqslant \frac{1}{t}(f - P_t f, f) \uparrow D(f)$, as $t \downarrow 0$,

$\exists \eta$: $\eta(r)/r \uparrow 1$ as $r \downarrow 0$: For f with $\pi(f) = 0$,

$$\|f\|^2 - \eta(t)D(f) \leqslant (P_t f, f) \leqslant \|P_t f\|\,\|f\|$$
$$\leqslant \sqrt{CV(f)\xi(t)}\,\|f\| \text{ (by assumption).}$$

$\Longrightarrow$

$$\|f\| \leqslant \frac{1}{2}\Big[\sqrt{CV(f)\xi(t)} + \sqrt{CV(f)\xi(t) + 4\eta(t)D(f)}\Big]$$

$\Longrightarrow \|f\|^2 \leqslant \eta(t)D(f) + C'V(f)\xi(t)$.

$\xi(t) = t^{1-q} (q > 1)$, $\eta(t) = t$, optimizing RHS w.r.t. t

$\Longrightarrow$

Liggett-Stroock : $\|\bar{f}\|^2 \leqslant C'' D(f)^{1/p} V(f)^{1/q}$.

2. Two functional inequalities

$\|\bar{f}\|^2 \leqslant \eta(t)D(f) + C'V(f)\xi(t), \quad \forall t > 0.$

Let $V(f) = 1$, RHS $= \eta(t)D(f) + C'\xi(t)$.

Define $\Phi(x) = \inf_{r>0}[\eta(r)x + C'\xi(r)]$. Then

$\|\bar{f}\|^2 \leqslant \Phi(D(f)), \qquad V(f) = 1.$

Not practical, since Φ is not explicit.

Regarding t parameter r. $\eta(t) = t$, $V(f) = \pi(|f|)^2$.

F. Y. Wang (2000), super-Poincaré inequality:

$\|f\|^2 \leqslant rD(f) + \beta(r)\pi(|f|)^2$, $\forall r > 0$. $\beta(r) \downarrow, r \uparrow$

$\Longleftrightarrow$ F-Sobolev inequality :

$\int f^2 F(f^2)\mathrm{d}\pi \leqslant CD(f)$, $\|f\| = 1$,

$\sup_{r\in(0,1]} |rF(r)| < \infty$, $\lim_{r\to\infty} F(r) = \infty$.

If $V = \|\cdot\|_\infty^2$, $\eta(r) = r$, then $\lim_{r\to 0}\beta(r) = \infty$. Use of parameter for convenience. RHS plays role mainly at r_0 only. Different pair (η, ξ) may achieve same value. Exchange.

F. Y. Wang and M. Röckner (2001), weaker-Poincaré inequality (WPI):

$$\|\bar{f}\|^2 \leqslant \alpha(r)D(f) + rV(f), \qquad \forall r > 0,$$

where $\alpha(r) \geqslant 0$, $\alpha(r) \downarrow$ as $r \uparrow$ on $(0, \infty)$.

When $V = \|\cdot\|_\infty^2$, equiv Kusuoka-Aida's WSGP: $\forall \{f_n\} \subset \mathscr{D}(D) : \pi(f_n) = 0, \|f_n\| \leqslant 1,$ $\lim_{n\to\infty} D(f_n) = 0 \Longrightarrow f_n \to 0$ in $\mathbb{P}$.

Theorem [F. Y. Wang and M. Röckner, 2001].

- If $\|P_t\bar f\|^2 \leqslant \xi(t)V(f), \forall t > 0, \xi(t) \downarrow 0$ as $t \uparrow \infty$, then WPI holds with the same V and

$$\alpha(r) = 2r \inf_{s>0} \frac{1}{s}\xi^{-1}\big(se^{1-s/r}\big),$$
$$\xi^{-1}(t) := \inf\{r > 0 : \xi(r) \leqslant t\}.$$

- If WPI holds and $V(P_t f) \leqslant V(f), \forall t \geqslant 0$, then

$$\|P_t\bar f\|^2 \leqslant \xi(t)\big[V(f) + \|\bar f\|^2\big], \quad \forall t > 0$$

where $\xi(t) = \inf\{r > 0 : -\alpha(r)\log r/2 \leqslant t\}$.

Cheeger's method.

3. Algebraic convergence (L.S.-inequality)

Example [C. and Y. Z. Wang, 2000]. $a_i = b_i = i^\gamma$, $i \gg 1$, $\gamma > 0$. Ergodic iff $\gamma > 1$.

- $\gamma > 1$. $\lambda_1 > 0$ iff $\gamma \geqslant 2$. W.r.t. $V(f) = \|\bar{f}\|^2$, L^2-algebraic decay iff $\gamma \geqslant 2$.
- $\gamma \in (1, 2)$. With respect to V^s: $V^s(f) = \sup_{k\geqslant 0} \big[(k+1)^s |f_{k+1} - f_k|\big]^2, 0 < s \leqslant \gamma - 1$ L^2-algebraic decay iff $\gamma \in (5/3, 2)$.
- $\gamma \in (1, 2)$. W.r.t. V_0: $V_0(f) = \sup_{i\neq j}(f_i - f_j)^2$, L^2-algebraic decay $\forall \gamma \in (1, 2)$.

Example [C. and Y. Z. Wang, 2000]. $a_i = 1$, $b_i = 1 - \gamma/i$, $i \gg 1$, $\gamma > 0$. Ergodic iff $\gamma > 1$.

- $\lambda_1 = 0$, $\forall \gamma > 1$.
- With respect to V^0: $V^0(f) = \sup_{k \geqslant 0} |f_{k+1} - f_k|^2$, L^2-algebraic decay iff $\gamma > 3$.
- With respect to V_0: $V_0(f) = \sup_{i \neq j}(f_i - f_j)^2$, L^2-algebraic decay $\forall \gamma > 1$.

Diffusions in $\mathbb{R}^d$ or on mfd. Y. Z. Wang, 1999.

One key point

$$V^s, V_0: \quad \mathrm{Lip}_\rho(f)^2 = \sup_{x\neq y}\left|\frac{f(x)-f(y)}{\rho(x,y)}\right|^2.$$

Liggett-Stroock : $\|\bar{f}\|^2 \leqslant CD(f)^{1/p}V(f)^{1/q}$

$$\mathrm{Var}(f) = \frac{1}{2}\sum_{i,j}\pi_i\pi_j(f_j-f_i)^2 = \sum_{\{i,j\}}\pi_i\pi_j(f_j-f_i)^2$$

$$\leqslant \left\{\sum_{\{i,j\}}\pi_i\pi_j\left(\frac{f_j-f_i}{\phi_{ij}}\right)^2\right\}^{1/p} \times$$

$$\left\{\sum_{\{i,j\}}\pi_i\pi_j\left(\frac{f_j-f_i}{\phi_{ij}^\delta}\right)^2\phi_{ij}^{2(q+\delta-1)}\right\}^{1/q}$$

Ergodic Markov Chain: $\pi_j := \lim_{t\to\infty} p_{ij}(t) > 0.$

Define $d_{ij}^{(n)} = \int_0^\infty t^n \big(p_{ij}(t) - \pi_j\big)\mathrm{d}t,\ \ n \in \mathbb{Z}_+,$

$m_{ij}^{(n)} = \mathbb{E}_i \sigma_j^n,\quad \sigma_j = \inf\{t \geqslant \tau_1 : X_t = j\}.$

Theorem [Y. H. Mao, 2002].

- $\left|d_{ij}^{(n)}\right| < \infty (\forall i, j)$ iff $m_j^{(n)} < \infty$ for some (all) j.
- $m_j^{(n)} < \infty \Longrightarrow p_{ij}(t) - \pi_j = o\big(t^{-n+1}\big)$, as $t \to \infty$.
- $m_j^{(n)} < \infty$ iff the inequalities

$$\begin{cases} \sum_k q_{ik} y_k \leqslant -n\, m_{ij}^{(n-1)}, & i \neq j \\ \sum_{k\neq j} q_{jk} y_k < \infty \end{cases}$$

have a finite non-negative solution (y_i). Equality.

4. General (irreversible) case.

Open problem. Criterion for slower convergence of general time-continuous Markov processes?

Time-discrete case. $(E, \mathscr{E})$. Process $(X_n)_{n\geqslant 0}$.
$\sigma_B = \inf\{n \geqslant 1 : X_n \in B\}$— Return time.

$$\mathscr{R}_0 = \left\{ r(n)_{n\in\mathbb{Z}_+} : 2 \leqslant r(n) \uparrow, \frac{\log r(n)}{n} \downarrow 0 \text{ as } n \uparrow \infty \right\}$$

$$\mathscr{R} = \left\{ r(n)_{n\in\mathbb{Z}_+} : \exists r_0 \in \mathscr{R}_0 \text{ such that} \right.$$

$$\left. \varliminf_{n\to\infty} \frac{r(n)}{r_0(n)} > 0 \text{ and } \varlimsup_{n\to\infty} \frac{r(n)}{r_0(n)} < \infty \right\}$$

Theorem [P. Tuominen and R. L. Tweedie, 1994]. Let irreducible and aperiodic. Given $r \in \mathscr{R}$. Then $\lim_{n\to\infty} r(n)\|P_n(x,\cdot)-\pi\|_{\text{Var}} = 0$ for all x in the set $\left\{x : \mathbb{E}_x \sum_{k=0}^{\sigma_B-1} r(k) < \infty,\ \forall B \in \mathscr{E}^+\right\}$, provided one of equivalent conditions holds:

- $\exists$ petite K: $\mathbb{E}_x \sum_{k=0}^{\sigma_K-1} r(k) < \infty,\ \forall x \in K$.
- $\exists (f_n)_{n\in\mathbb{Z}_+}$: $E \to [1,\infty]$, petite K and const b: $\sup_{x\in K} f_0 < \infty$, $\{f_1 < \infty\} \subset \{f_0 < \infty\}$ and $P f_{n+1} \leqslant f_n - r(n) + b r(n) I_K,\ n \in \mathbb{Z}_+$.
- $\exists A \in \mathscr{E}^+$ such that $\sup_{x\in A} \mathbb{E}_x \sum_{k=0}^{\sigma_B-1} r(k) < \infty,\ \forall B \in \mathscr{E}^+$.

S. F. Jarner & G. Roberts (2000): Poly, single f.

A Diagram of Nine Types of Ergodicity

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Stochastic analysis in infinite dimensional spaces

RIMS, Kyoto University, Japan
(November 6–8, 2002)[8]

Contents

- Ergodicity by means of three inequalities.
- Three traditional types of ergodicity.
- Diagram of nine types of ergodicity.
- Applications and comments.
- Partial proofs of the diagram.

1. Ergodicity by means of three inequalities

Three basic inequalities

Prob. space $(E, \mathscr{E}, \pi)$. Dirichlet form $(D, \mathscr{D}(D))$.

Poincaré inequality : $\mathrm{Var}(f) \leqslant \lambda_1^{-1} D(f)$

Nash inequality : $\mathrm{Var}(f)^{1+2/\nu} \leqslant \eta^{-1} D(f) \|f\|_1^{4/\nu}$

(J. Nash, 1958) $\quad \nu > 0$

Equivalent if $\|f\|_1 \longrightarrow \|f\|_r$ for all $r \in (1, 2)$.

Logarithmic Sobolev inequality (L. Gross, 1976) :

Log Sobolev : $\int f^2 \log\left(f^2/\|f\|^2\right) \mathrm{d}\pi \leqslant 2\sigma^{-1} D(f)$.

Ergodicity by means the inequalities

$(D, \mathscr{D}(D)) \longrightarrow$ semigroup (P_t): $P_t = e^{tL}$

Theorem [T.M.Liggett(89), L.Gross(76), C.(99)]

- Poincaré ineq. $\Longleftrightarrow \mathrm{Var}(P_t f) \leqslant \mathrm{Var}(f) e^{-2\lambda_1 t}$.
- LogS $\Longrightarrow$ exponential convergence in entropy:
$$\mathrm{Ent}(P_t f) \leqslant \mathrm{Ent}(f) e^{-2\sigma t},$$
where $\mathrm{Ent}(f) = \pi(f \log f) - \pi(f) \log \|f\|_1$.
- Nash ineq. $\Longleftrightarrow \mathrm{Var}(P_t f) \leqslant \left(\frac{\nu}{4\eta t}\right)^{\nu/2} \|f\|_1^2$.

Theorem (L. Gross) :

LogS $\Longrightarrow$ Exp. conv. in entropy $\Longrightarrow$ Poincaré ineq.

Nash inequality weakest?

2. Three traditional types of ergodicity.

$$\|\mu-\nu\|_{\text{Var}} = 2\sup_{A\in\mathscr{E}}|\mu(A)-\nu(A)|$$

Ordinary erg. : $\lim_{t\to\infty}\|P_t(x,\cdot)-\pi\|_{\text{Var}}=0$

Exp. erg. : $\lim_{t\to\infty}e^{\hat{\alpha}t}\|P_t(x,\cdot)-\pi\|_{\text{Var}}=0$

Strong erg. : $\lim_{t\to\infty}\sup_x\|P_t(x,\cdot)-\pi\|_{\text{Var}}=0$

$$\Longleftrightarrow \lim_{t\to\infty}e^{\hat{\beta}t}\sup_x\|P_t(x,\cdot)-\pi\|_{\text{Var}}=0$$

Classical Theorem :

Strong ergodicity $\Longrightarrow$ Exp. erg. $\Longrightarrow$ Ordinary erg.

Example: Two points

$$Q = \begin{pmatrix} -b & b \\ a & -a \end{pmatrix}. \qquad P(t) = e^{tQ}.$$

$$P(t) = \frac{1}{a+b}\begin{pmatrix} a + be^{-\lambda_1 t} & b\left(1 - e^{-\lambda_1 t}\right) \\ a\left(1 - e^{-\lambda_1 t}\right) & b + ae^{-\lambda_1 t} \end{pmatrix}.$$

$\lambda_1 = \hat{\alpha} = \hat{\beta}$	Log Sobolev σ	Nash η
$a + b \geqslant$	$\frac{2(a\vee b - a\wedge b)}{\log a \vee b - \log a \wedge b}$	$(a+b)\left(\frac{a\wedge b}{a\vee b}\right)^{1+2/\nu}$

Question: Exact rate of exp. conv. in entropy?

$$\|\mu_1 - \mu_2\|_{\text{Var}} = \inf_{\tilde{\mu}} \int_{E\times E} d(x_1, x_2)\tilde{\mu}(\mathrm{d}x_1, \mathrm{d}x_2).$$

3. Diagram of ergodicity

Theorem. For revers. Markov proc. with density,

Nash inequality

$\swarrow\!\!\swarrow$ $\searrow\!\!\searrow$

LogS inequality $\qquad$ L^1-exp. convergence

$\Downarrow$ $\qquad$ $\|$

Exp. conv. in entropy $\qquad$ π-a.s. Strong ergodicity

$\Downarrow$ $\qquad$ $\Downarrow$

Poincaré inequality $\Longleftrightarrow$ π-a.s. Exp. ergodicity

$\Downarrow$

L^2-algebraic ergodicity

$\Downarrow$

Ordinary ergodicity

4. Applications and comments.

- Explicit criterion for exp. erg. in dim. one.
- Criterion for L^2-spectral gap. Operator L, reversible, irreducible, aperiodic,

$$LV \leqslant (-\delta + ch)V, \quad R_1 \geqslant h \otimes \nu,$$
$$V : E \to [1, \infty], h : E \to (0, 1], \delta > 0, c < \infty, \nu$$

- Criterion (explicit in dim. one) for L^1-sp. gap.
- Reversibility is used in two places.
- Density is used only in the identity.
- Complete.

Examples: Comparisons by diffusions on $[0, \infty)$

	Erg.	Exp.erg.	LogS	Str.erg.	Nash
$b(x) = 0$ $a(x) = x^\gamma$	$\gamma > 1$	$\gamma \geqslant 2$	$\gamma > 2$	$\gamma > 2$	$\gamma > 2$
$b(x) = 0$ $a(x) = x^2 \log^\gamma x$	$\surd$	$\gamma \geqslant 0$	$\gamma \geqslant 1$	$\gamma > 1$	$\times$
$a(x) = 1$ $b(x) = -b$	$\surd$	$\surd$	$\times$	$\times$	$\times$

Examples: Comparisons by birth-death processes

	Erg.	Exp.erg.	LogS	Str.erg.	Nash
$a_i = b_i = i^\gamma$	$\gamma > 1$	$\gamma \geqslant 2$	$\gamma > 2$	$\gamma > 2$	$\gamma > 2$
$a_i = b_i = i^2 \log^\gamma i$	$\surd$	$\gamma \geqslant 0$	$\gamma \geqslant 1$	$\gamma > 1$	$\times$
$a_i = a >$ $b_i = b$	$\surd$	$\surd$	$\times$	$\times$	$\times$

Example (C., 99): LogS and strong ergodicity are not comparable!

Let $(\pi_i > 0)$ and take $q_{ij} = \pi_j (j \neq i)$.

- LogS does not hold: (q_{ij}) is bounded.
- Strongly ergodic: $y_0 = 0$, $y_i = 1/\pi_0 (i \neq 0)$ satisfy the inequalities' criterion

$$\begin{cases} \sum_j q_{ij}(y_j - y_i) \leqslant -1, & i \neq 0 \\ \sum_{j\neq 0} q_{0j} y_j < \infty, & (y_i) \text{ bdd }. \end{cases}$$

Example (F. Y. Wang, 2001): One dim. diffusion.

Eleven criteria for one-dimensional diffusions

Diffusion processes on $[0, \infty)$

Reflecting boundary at 0.

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}. \quad C(x) = \int_0^x b/a,$$

$$\mathrm{d}\mu = \frac{e^C}{a}\mathrm{d}x, \quad \mu[x, y] = \int_x^y e^C/a.$$

Property	Criterion
Uniq.	$\int_0^\infty \mu[0,x]e^{-C(x)} = \infty \quad (*)$
Recur.	$\int_0^\infty e^{-C(x)} = \infty$
Ergod.	$(*)\ \&\ \mu[0,\infty) < \infty$
Exp. Erg. Poinc.	$(*)\ \&\ \sup_{x>0} \mu[x,\infty) \int_0^x e^{-C} < \infty$
Dis. Sp.	$(*)\ \&\ \lim_{n\to\infty} \sup_{x>n} \mu[x,\infty) \int_n^x e^{-C} = 0$
LogS Exp. ent.	$(*)\ \&\sup_{x>0} \mu[x,\infty)\log[\mu[x,\infty)^{-1}]\int_0^x e^{-C} < \infty$
Str. Erg. L^1-Exp.	$(*)\ \&\ \int_0^\infty \mu[x,\infty)e^{-C(x)} < \infty$
Nash	$(*)\ \&\ \sup_{x>0} \mu[x,\infty)^{(\nu-2)/\nu} \int_0^x e^{-C} < \infty (\varepsilon)$

5. Partial proofs of diagram

(a) Nash inequality$\Longrightarrow$ L^1-exponential convergence and π-a.s. strong ergodicity.

(b) L^1-exponential convergence $=$ π-a.s. strong ergodicity.

(c) Nash inequality $\Longrightarrow$ logarithmic Sobolev inequality.

(d) L^2-exponential convergence$\Longrightarrow$exponential ergodicity.

(e) Exponential ergodicity$\Longrightarrow$ L^2-exponential convergence.

(a) Nash inequality $\Longrightarrow L^1$-exponential convergence & π-a.s. strong ergodicity [C. (1999)].

Nash inequality $\Longleftrightarrow \|P_t - \pi\|_{1\to 2}^2 \leqslant C/t^{q-1}$.

Nash inequality $\qquad q := 1 + \nu/2$

$$\Longleftrightarrow \mathrm{Var}(P_t(f)) = \|P_t f - \pi(f)\|_2^2 \leqslant C^2 \|f\|_1^2 / t^{q-1}$$

$$\Longleftrightarrow \|(P_t - \pi)f\|_2 \leqslant C\|f\|_1 / t^{(q-1)/2}.$$

$$\Longleftrightarrow \|P_t - \pi\|_{1\to 2} \leqslant C/t^{(q-1)/2}.$$

Operator norm : $\|\cdot\|_{p\to q},\ L^p(\pi) \to L^q(\pi)$.

- Nash inequality $\Longrightarrow L^1$-exp. convergence.

$\|P_t - \pi\|_{1\to 1} \leqslant \|P_t - \pi\|_{1\to 2}$. Algebraic, exp.

- Nash inequality $\Longrightarrow$ π-a.s. strong ergodicity.

Symmetry : $P_t - \pi = (P_t - \pi)^*$.

$$\|P_{2t} - \pi\|_{1\to\infty} \leqslant \|P_t - \pi\|_{1\to 2}\|P_t - \pi\|_{2\to\infty}$$
$$= \|P_t - \pi\|_{1\to 2}\|(P_t - \pi)^*\|_{1\to 2} = \|P_t - \pi\|_{1\to 2}^2.$$

$$\begin{aligned}
&\text{ess sup}_x \|P_t(x,\cdot) - \pi\|_{\text{Var}} \\
&= \text{ess sup}_x \sup_{|f|\leqslant 1} |(P_t(x,\cdot) - \pi) f| \\
&\leqslant \text{ess sup}_x \sup_{\|f\|_1\leqslant 1} |(P_t(x,\cdot) - \pi) f| \\
&= \sup_{\|f\|_1\leqslant 1} \text{ess sup}_x |(P_t(x,\cdot) - \pi) f| \\
&= \|P_t - \pi\|_{1\to\infty} \leqslant C/t^{q-1} \to 0.
\end{aligned}$$

(b) L^1-exponential convergence $=$ π-a.s. strong ergodicity [Y.H.Mao(2002)]

$$(L^1)^* = L^\infty \Longrightarrow \|P_t - \pi\|_{1\to 1} = \|P_t^* - \pi\|_{\infty\to\infty}.$$

$$\begin{aligned}
&\|P_t^* - \pi\|_{\infty\to\infty}\\
&\quad = \text{ess sup}_x \sup_{\|f\|_\infty = 1} |(P_t^* - \pi) f(x)|\\
&\quad = \text{ess sup}_x \sup_{\sup |f| = 1} |(P_t^* - \pi) f(x)|\\
&\quad (\text{since } P_t^*(x,\cdot) \ll \pi. \text{ Otherwise } \geqslant)\\
&\quad = \text{ess sup}_x \|P_t^*(x,\cdot) - \pi\|_{\text{Var}}.
\end{aligned}$$

L^1-exp. con. $=$ ($\Rightarrow$) π-a.s. strong ergodicity.

Question: Remove hypothesis on density?

(c) Nash inequality $\Longrightarrow$ logarithmic Sobolev inequality [C. (1999)].

Because $\|f\|_1 \leqslant \|f\|_p$ for all $p \geqslant 1$,
$\|\cdot\|_{2\to 2} \leqslant \|\cdot\|_{1\to 2} \leqslant C/t^{(q-1)/2}$.
Nash inequality$\Longrightarrow$Poincaré inequality$\Longleftrightarrow$ $\lambda_1 > 0$.
$\|P_t\|_{p\to 2} \leqslant \|P_t\|_{1\to 2} \leqslant \|P_t - \pi\|_{1\to 2} + \|\pi\|_{1\to 2} < \infty$,

$$p \in (1, 2).$$

Bakry's survey article (1992): Theorem 3.6 and Proposition 3.9.

(d) L^2-exponential convergence$\Longrightarrow$ π-a.s. exponential ergodicity[C. (1991, 1998, 2000)].

Let $\mu \ll \pi$, then

$$\|\mu P_t - \pi\|_{\text{Var}} = \sup_{|f|\leqslant 1} |(\mu P_t - \pi) f|$$

$$= \sup_{|f|\leqslant 1} \left|\pi\left(\frac{\mathrm{d}\mu}{\mathrm{d}\pi} P_t f - f\right)\right| = \sup_{|f|\leqslant 1} \left|\pi\left(f P_t^*\left(\frac{\mathrm{d}\mu}{\mathrm{d}\pi}\right) - f\right)\right|$$

$$= \sup_{|f|\leqslant 1} \left|\pi\left[f\left(P_t^*\left(\frac{\mathrm{d}\mu}{\mathrm{d}\pi} - 1\right)\right)\right]\right|$$

$$\leqslant \left\|P_t^*\left(\frac{\mathrm{d}\mu}{\mathrm{d}\pi} - 1\right)\right\|_1 \leqslant \left\|\frac{\mathrm{d}\mu}{\mathrm{d}\pi} - 1\right\|_2 e^{-t\,\mathbf{gap}(L^*)}$$

$$= \left\|\frac{\mathrm{d}\mu}{\mathrm{d}\pi} - 1\right\|_2 e^{-t\,\mathbf{gap}(L)}.$$

Reversible case: Need density $P_t(x,\cdot) \ll \pi$!

Suppose $p_t(x,y) := \frac{\mathrm{d}P_t(x,\cdot)}{\mathrm{d}\pi}(y)$. $p_t(x,y) = p_t(y,x)$.

$$\int p_s(x,y)^2\pi(\mathrm{d}y) = \int p_s(x,y)p_s(y,x)\pi(\mathrm{d}y)$$
$$= p_{2s}(x,x) < \infty,\ \pi\text{-a.s.}(x).$$

$$\|P_t(x,\cdot)-\pi\|_{\mathrm{Var}} \leqslant \left[\sqrt{p_{2s}(x,x)-1}\,e^{s\,\lambda_1}\right]e^{-t\,\lambda_1}, t \geqslant s.$$

$$\exists C(x)<\infty, \|P_t(x,\cdot)-\pi\|_{\mathrm{Var}} \leqslant C(x)e^{-t\,\lambda_1}, \quad t \geqslant 0.$$

Largest ε_1: $\|\|P_t(\bullet,\cdot) - \pi\|_{\mathrm{Var}}\|_1 \leqslant Ce^{-\varepsilon_1 t}$.

If $p_{2s}(\cdot,\cdot) \in L^{1/2}(\pi)$, then $\varepsilon_1 \geqslant \mathrm{gap}(L) = \lambda_1$.

φ-irreducible case: Take $\mathrm{d}\mu = \frac{I_B}{\pi(B)}\mathrm{d}\pi$, small B.

(e) π-a.s. exponential ergodicity $\Longrightarrow$ L^2-exponential convergence [C. (2000), Mao (2002)]

- π-a.s. exponential ergodicity $\iff$ $\big\| \|P_t(\bullet,\cdot) - \pi\|_{\text{Var}}\big\|_1$ exponential convergence.
- $\big\| \|P_t(\bullet,\cdot) - \pi\|_{\text{Var}}\big\|_1 \geqslant \|P_t - \pi\|_{\infty\to 1}$.
- $\|P_t - \pi\|_{\infty\to 1} \geqslant \|P_t - \pi\|_{\infty\to 2}^2/2$.
- $\lambda_1 = \text{gap}(L) \geqslant \varepsilon_1/2$.

Prove that π-a.s. exponential ergodicity

$\iff \|\|\|P_t(\bullet,\cdot)-\pi\|_{\text{Var}}\|_1$ exponential convergence.

Time-discrete case was proved by G. O. Roberts & J. S. Rosenthal(97). Let $\mathscr{E}$ be countably generated.

E. Numemelin & P. Tuominen (1982), Stoch. Proc. Appl.

π-a.s. geometric ergodicity

$\iff \|\|\|P^n(\bullet,\cdot) - \pi\|_{\text{Var}}\|_1$ geometric convergence.

Time-discrete$\Longrightarrow$Time-continuous

(C.'s book, §4.4]).

π-a.s. exponential ergodicity

$\iff \|\|\|P_t(\bullet,\cdot)-\pi\|_{\text{Var}}\|_1$ exponential convergence.

Assume that $\|\|\|P_t(\bullet,\cdot) - \pi\|_{\text{Var}}\|_1 \leqslant Ce^{-\varepsilon_1 t}$.

Prove that $\| \| P_t(\bullet, \cdot) - \pi \|_{\mathrm{Var}} \|_1 \geqslant \| P_t - \pi \|_{\infty \to 1}$.

Let $\| f \|_\infty = 1$. Then

$$\| (P_t - \pi) f \|_1 = \int \pi(\mathrm{d}x) \left| \int \left[P_t(x, \mathrm{d}y) - \pi(\mathrm{d}y) \right] f(y) \right|$$

$$\leqslant \int \pi(\mathrm{d}x) \sup_{\|g\|_\infty \leqslant 1} \left| \int \left[P_t(x, \mathrm{d}y) - \pi(\mathrm{d}y) \right] g(y) \right|$$

$$= \| \| P_t(\bullet, \cdot) - \pi \|_{\mathrm{Var}} \|_1$$

(Need $P_t(x, \cdot) \ll \pi$ or reversibility!).

Prove that $\|P_{2t} - \pi\|_{\infty\to 1} = \|P_t - \pi\|^2_{\infty\to 2}$.

$$\begin{aligned}\|(P_t - \pi)f\|_2^2 &= ((P_t - \pi)f, (P_t - \pi)f) \\ &= (f, (P_t - \pi)^2 f) \\ &= (f, (P_{2t} - \pi)f) \\ &\leqslant \|f\|_\infty \|(P_{2t} - \pi)f\|_1 \\ &\leqslant \|f\|_\infty^2 \|P_{2t} - \pi\|_{\infty\to 1}.\end{aligned}$$

$\|P_{2t} - \pi\|_{\infty\to 1} \geqslant \|P_t - \pi\|^2_{\infty\to 2}$. Inverse obvious. Need reversibility. Otherwise, we have $\|P_t - \pi\|_{\infty\to 1} \geqslant \|P_t - \pi\|^2_{\infty\to 2}/2$.

Prove that $\lambda_1 = \text{gap}(L) \geqslant \varepsilon_1$

$$\|(P_t - \pi)f\|_2^2 \leqslant C\|f\|_\infty^2 e^{-2\varepsilon_1 t}.$$

$$\|P_t f\|_2^2 \leqslant C\|f\|_\infty^2 e^{-2\varepsilon_1 t}, \; \pi(f) = 0, \|f\|_2 = 1.$$

(Wang(2000)) :

$$\|P_t f\|_2^2 = \int_0^\infty e^{-2\lambda t} \mathbf{d}(E_\lambda f, f) \quad (\text{Reversiblity!})$$

$$\geqslant \Big[\int_0^\infty e^{-2\lambda s} \mathbf{d}(E_\lambda f, f)\Big]^{t/s}, \; t \geqslant s$$

(Jensen inequality)

$$= \|P_s f\|_2^{2t/s}.$$

$$\|P_s f\|_2^2 \leqslant \Big[C\|f\|_\infty^2\Big]^{s/t} e^{-2\varepsilon_1 s}.$$

$$\|P_s f\|_2^2 \leqslant e^{-2\varepsilon_1 s} \ (t \to \infty),$$

$$\pi(f) = 0,\ \|f\|_2 = 1,\ f \in L^\infty(\pi).$$

$$\|P_s f\|_2^2 \leqslant e^{-2\varepsilon_1 s}, \quad s \geqslant 0,$$

$$\pi(f) = 0,\ \|f\|_2 = 1 \text{ (dense)}.$$

$$\lambda_1 \geqslant \varepsilon_1.$$

Reversible case.

- If $p_t(\cdot,\cdot)\in L^{1/2}(\pi)$ for some $t>0$, then $\lambda_1 = \varepsilon_1$.
- Wang (2002):

$$\sup\left\{\|\mu P_t-\pi\|_{\text{Var}}\Big/\left\|\frac{\mathrm{d}\mu}{\mathrm{d}\pi}-1\right\|_2 : \mu\ll\nu\right\}\sim e^{-t\,\lambda_1}.$$

- Time-discrete reversible Markov processes.
  G. O. Roberts and J. S. Rosenthal (1997),
  G. O. Roberts and R. L. Tweedie (2001).

Problems: Irreversible processes.
Infinitely dimensional situation.

Reaction–Diffusion Processes

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Contents

IPS, Random Fields: Re-establish foundation of Statistical Mechanics.

Equilibrium systems: Ising model.

Reaction-diffusion processes, RD-processes.

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Hard and challenge: ∞-dimensional mathematics.

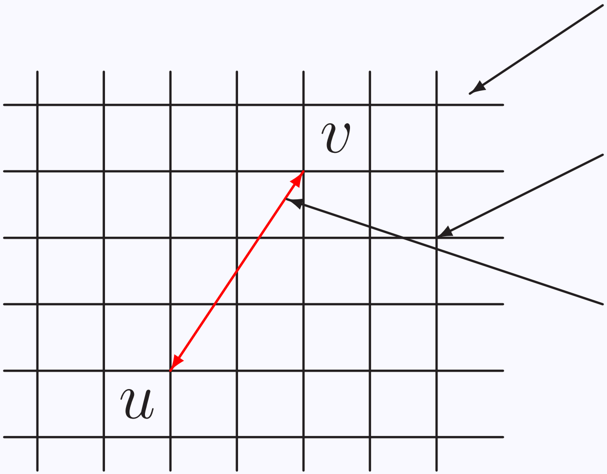
Re-examine math tools in finite-dimensional math.

Develop, or create new mathematical tools.

Turn to apply to probability theory & other branches of mathematics.

Open problems.

1. The models.



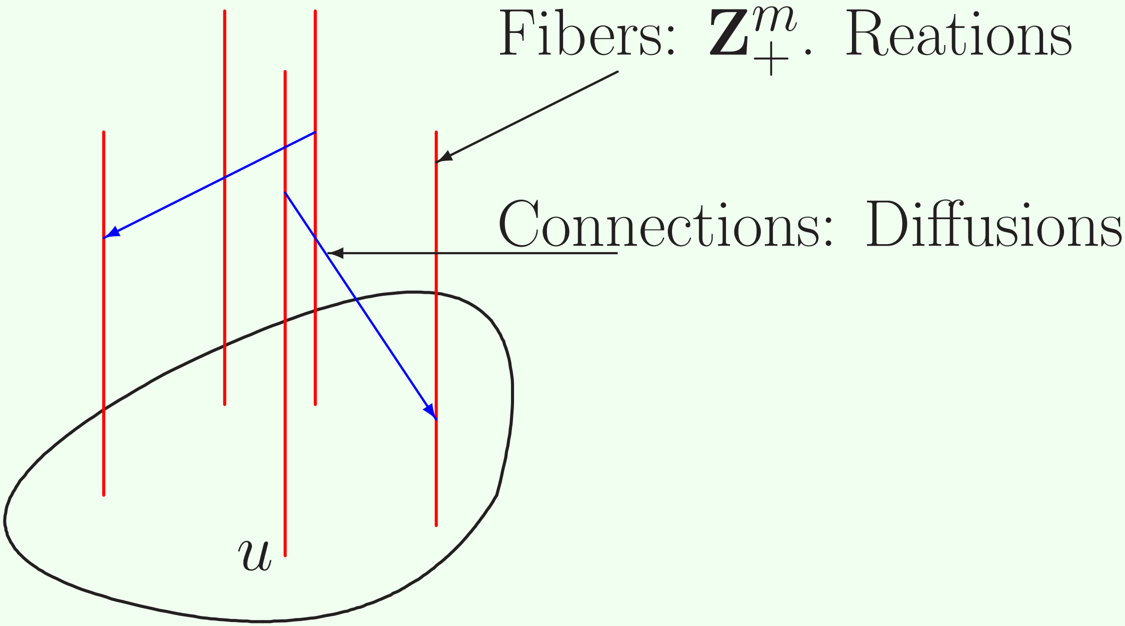
Container $S = \mathbf{Z}^d$

Small vessels: reactions

Diffusions

State space $E = \left(\mathbf{Z}_+^m\right)^S$

Informal interpretation: Dynamics of ∞-dim fiber bundle



Base space: $S = \mathbf{Z}^d$

Reaction in small vessel $u \in S$. # of particles. Single reactant: Q-matrices $Q_u = (q_u(i,j) : i,j \in \mathbf{Z}_+)$. Totally stable & conservative:

$$-q_u(i,i) = \sum_{j \neq i} q_u(i,j) < \infty, \quad \forall i \in \mathbf{Z}_+.$$

Reaction part of the operator:

$$\Omega_r f(x) = \sum_{u \in S} \sum_{k \in \mathbf{Z} \setminus \{0\}} q_u(x_u, x_u + k) \big[f(x + k e_u) - f(x) \big],$$

e_u unit vector in $E := \mathbf{Z}_+^S$, convention $q_u(i,j) = 0$ for $i \in \mathbf{Z}_+$, $j \notin \mathbf{Z}_+$, $u \in S$.

Diffusion between the vessels, transition probability $(p(u,v) : u, v \in S)$ & $c_u\,(u \in S)$ on $\mathbf{Z}_+$. If k particles in u, then rate from u to v is $c_u(k)p(u,v)$. c_u satisfies

$$c_u \geqslant c_u(0) = 0, \qquad u \in S.$$

Diffusion part of the operator:

$$\Omega_d f(x) = \sum_{u,v\in S} c_u(x_u)p(u,v)[f(x - e_u + e_v) - f(x)]$$

Whole operator: $\Omega = \Omega_r + \Omega_d$.

Example 1 (Polynomial model). Diffusion rates:

$c_u(k) = k, \quad p(u,v)$: simple RW on $\mathbf{Z}^d$.

Reaction rates: birth-death

$$q_u(k, k+1) = b_k = \sum_{j=0}^{m_0} \beta_j k^{(j)},$$

$$q_u(k, k-1) = a_k = \sum_{j=1}^{m_0+1} \delta_j k^{(j)},$$

$$k^{(j)} = k(k-1)\cdots(k-j+1),$$

$$\beta_j, \delta_j \geqslant 0; \quad \beta_0, \beta_{m_0}, \delta_1, \delta_{m_0+1} > 0.$$

Example 2. (1) Schlögl's 1'st model: $m_0 = 1$.

$$b_k = \beta_0 + \beta_1 k, \quad a_k = \delta_1 k + \delta_2 k(k-1).$$

(2) Schlögl's 2'nd model: $m_0 = 2$ but $\beta_1 = \delta_2 = 0$.

$$b_k = \beta_0 + \beta_2 k(k-1),\ a_k = \delta_1 k + \delta_3 k(k-1)(k-2).$$

Example 3 (Brussel's model). Two types of particles. For each type, diffusion part is same. Reaction part

$$\begin{aligned}(i,\ j) &\to (i+1,\ j) && \text{at rate} \quad \lambda_1\\ &\to (i-1,\ j) && \text{at rate} \quad \lambda_4 i\\ &\to (i-1,\ j+1) && \text{at rate} \quad \lambda_2 i\\ &\to (i+1,\ j-1) && \text{at rate} \quad \lambda_3 i^{(2)} j, \qquad \lambda_k > 0\end{aligned}$$

Typical models in non-equilibrium statistical physics. 15 models.

2. Finite-dimensional case.

(a) Uniqueness criterion.

(b) Recurrence & positive recurrence.

Finite S. $E = \mathbf{Z}_+^S \left(\text{or } \left(\mathbf{Z}_+^2\right)^S\right)$. MCs.

(a) Uniqueness criterion. The process is unique (non-explosive) iff the equation

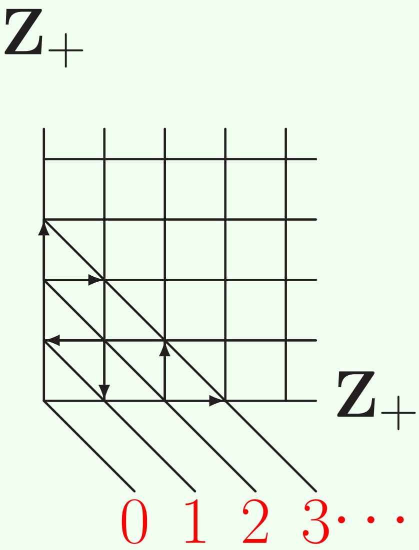
$$(\lambda I - \Omega)u(x) = 0, \; 0 \leqslant u(x) \leqslant 1, \; x \in \mathbf{Z}_+^S.$$

has zero-solution for some (equivalently, any) $\lambda > 0$.

Misleading. Does not care geometry.

C. & S. J. Yan (1986).

Reduce higher-dim to dim one



$E_n := \{x : \sum_{u \in S} x_u = n\} \longrightarrow n$

$q_{23} = \max_{y \in E_2} \sum_{z \in E_3} \text{Rate}(y \to z)$

$q_{21} = \min_{y \in E_2} \sum_{z \in E_1} \text{Rate}(y \to z)$

Single birth processes:

$q_{i,i+n} > 0$ iff $n = 1$

Theorem 1[C., 1986]. $Q = (q_{ij})$ on countable E.
Suppose $\exists \{E_n\}_1^\infty$, $\varphi \geqslant 0$ s.t.

- $E_n \uparrow E$, $\sup\limits_{i \in E_n} (-q_{ii}) < \infty$, $\lim\limits_{n\to\infty} \inf\limits_{i \notin E_n} \varphi_i = \infty$,
- $\exists c \in \mathbf{R}$ s.t. $\sum_j q_{ij}(\varphi_j - \varphi_i) \leqslant c\,\varphi_i$, $\forall i \in E$,

then MC unique.

$\varphi(x) = c[1 + \sum_u x_u]$, $\quad E_n = \{x : \sum_{u \in S} x_u \leqslant n\}$
C., 1986, 1991, 1992, 1997
Various application and extension: 35 papers.
W. J. Anderson's book, 1991
The first mathematical tool.

(b) Recurrence & positive recurrence.

Schlögl's: C. & S. J. Yan 1986.

Brussel's model:

Single vessel, D. Han (1990);

Finite vessels, J. W. Chen (1991).

$|S| < \infty$: No phase transitions. Go to ∞-dim.

Comparison of Ising model and Schlögl model

Comparison	Ising Model	Schlögl model
Space	$\{-1,+1\}^{\mathbf{Z}^d}$ compact	$\mathbf{Z}_+{}^{\mathbf{Z}^d}$: not locally compact
System	equilibrium reversible	non-equilibrium irreversible
Operator	locally bdd	not locally bdd & non-linear
Stationary distribution	always $\exists$ & locally explicit	? locally no expression

3. Construction of the processes.

No Ω_d: Independent product of Markov chains.
No Ω_r: zero range processes.

- R. Holley (1970),
- T. M. Liggett (1973),
- E. D. Andjel (1982),
- Liggett & F. Spitzer (1981), 6 models: locally bdd & linear.

Semigroup theory, Liggett (1972, 1985).
Weak convergence. Stronger convergence.

Given probabilities P_1 & P_2 on $(E, \mathscr{E})$, **coupling** of P_1 & P_2 is probability $\widetilde{P}$ on product space $(E^2, \mathscr{E}^2)$ **marginality**:

$$\widetilde{P}(A\times E)=P_1(A), \quad \widetilde{P}(E\times A)=P_2(A), \quad \forall A\in\mathscr{E}.$$

$(E, \rho, \mathscr{E})$: metric space with distance ρ.
Wasserstein distance $W(P_1, P_2)$:

$$W(P_1, P_2) = \inf_{\widetilde{P}} \int_{E^2} \rho(x_1, x_2)\widetilde{P}(dx_1, dx_2),$$

$\widetilde{P}$ varies over all couplings of P_1 and P_2.

Strategy: Take finite $\{\Lambda_n\}$, $\Lambda_n \uparrow S$.

MC $P_n(t, x, \cdot)$ on $E_n := \mathbf{Z}_+^{\Lambda_n}$.

Regard $P_n(t, x, \cdot)$ as a MC on E_m for $n < m$.

Fixed $t \geqslant 0$ & $x \in E_m$, $W(P_n(t, x, \cdot), P_m(t, x, \cdot))$ is well defined.

One key step in our construction is to prove that

$$W(P_n(t, x, \cdot),\ P_m(t, x, \cdot)) \longrightarrow 0 \text{ as } m, n \to \infty.$$

W-distance $\rightarrow$ coupling $\rightarrow$ Markov coupling, $P_n(t, x, \cdot) \rightarrow \Omega_n$. Theory of coupling for time-continuous Markov processes.

4. Markovian couplings. Original point. Given $P_k(t)$ on E_k $(k = 1, 2)$, construct $\widetilde{P}(t)$ on $E_1 \times E_2$

$$\widetilde{P}(t)\tilde{f}_k(x_1,x_2)=P_k(t)f_k(x_k), x_k \in E_k, k=1,2 \qquad \text{(MP)}$$

for all bdd f_k on E_k, $\tilde{f}_k(x_1, x_2) = f_k(x_k), k = 1, 2.$

$$Q = (q_{ij}) \leftrightarrow \Omega f(i) = \sum_{j \neq i} q_{ij}(f_j - f_i).$$

Operators Ω_1, Ω_2 & $\widetilde{\Omega}$ for $\widetilde{P}(t)$.

Marginality for operators: By (MP),

$$\widetilde{\Omega}\tilde{f}_k(x_1, x_2)=\Omega_k f_k(x_k), x_k \in E_k, k=1,2 \qquad \text{(MO)}$$

$\widetilde{\Omega}$: **coupling operator**.

Exists coupling operator? ∞. Really hard problem.

Theorem 2 [C., 1986]. If a coupling operator determines uniquely a MC (=non-explosive), then so do the marginals. Conversely, if each of the marginals determines uniquely a MC, then so does every coupling operator. Furthermore (MP) $\Longleftrightarrow$ (MO).

- MCs $\longrightarrow$ general Markov jump processes,
- discrete space $\longrightarrow$ continuous space,
- Markov coupling $\longrightarrow$ optimal Markov coupling,
- exponential convergence $\longrightarrow$ spectral gap,
- compact manifolds $\longrightarrow$ non-compact manifolds,
- finite-dimension $\longrightarrow$ ∞-dimension

C., 1992, 1994, 1997abcd, 2002. Second math tool.

5. Construction of processes (continued).

$k_u > 0, \ \sum_{u\in S} k_u < \infty. \quad \|x\| = \sum_{u\in S} x_u k_u;$

$E_0 = \{x \in E : \|x\| < \infty\}.$

Key estimates:

(1) $P_n(t)\|\cdot\|(x) \leqslant (1 + \|x\|)e^{ct}, \quad x \in E_0$

(2) $W_{\Lambda_n}(P_n(t, x, \cdot), P_m(t, x, \cdot)) \leqslant c(t, \Lambda_n, x; n, m),$

$x \in E_0, \quad c$: constant,

W_V: W-distance, $\mathbf{Z}_+^V$, w.r.t. $\sum_{u\in V} k_u|x_u - y_u|$,

$\lim_{m\geqslant n\to\infty} c(t, \Lambda_n, x; n, m) = 0.$

Assumptions. $c_u \geqslant c_u(0) = 0, \qquad u \in S.$

$$\sup_v \sum_u p(u,v) < \infty, \ \sup_{k,u} |c_u(k) - c_u(k+1)| < \infty.$$

$$\sum_{k\neq 0} q_u(i, i+k)|k| < \infty, \ u \in S.$$

$$\sup_{u\in S,\, j_2>j_1\geqslant 0} [g_u(j_1,j_2)+h_u(j_1,j_2)] < \infty,$$

$$g_u(j_1,j_2) = \frac{1}{j_2-j_1}\sum_{k\neq 0}\big[q_u(j_2, j_2+k) - q_u(j_1, j_1+k)\big]k,$$

$$h_u(j_1,j_2) = \frac{2}{j_2-j_1}\sum_{k=0}^{\infty}\Big[\big(q_u(j_2, j_1-k) - q_u(j_1, 2j_1-j_2-k)\big)^+ + \big(q_u(j_1, j_2+k) - q_u(j_2, 2j_2-j_1+k)\big)^+\Big]k, \ j_2 > j_1$$

Theorem 3 [C., 1985]. Under the assumptions,

- $\exists$ MP Lipschitz (P_t).
- For Lipschitz f on E_0, $\frac{d}{dt}P_t f\big|_{t=0} = \Omega f$ in a dense set of E_0.

Two essential conditions (colored).

Applicable to Examples 1 & 2.

For Example 3: Bdd c_u, S. Z. Tang (1985);

$c_u(k) \leqslant \log k$, D. Han (1990–1995), martingale approach.

Problem 1. Construct a MP for Brussel's model.

Theorem 4 [C., 1991; Y. Li, 1991]. Under same assumption of Theorem 3 if additionally

$$\sup_u \sum_{k\neq 0} q_u(i, i+k)\left[(i+k)^m - i^m\right] \leqslant c(1+i^m), \quad i\in \mathbf{Z}_+$$

for some $m > 1$, then process is also unique.

∞-dimensional version of maximal principle, S. Z. Tang & Y. Li.
The third mathematical tool.

6. Existence of stationary distributions.

Compact: $\exists$. Non-compact: case by case

Theorem 5 [C., 1986, 1989; L. P. Huang, 1987].

Always $\exists$ for polynomial model.

Order of death rate $\geqslant$ the birth one.

Proof depends heavily on the construction of the process.

7. Ergodicity.

(a) General case. (b) Reversible case.

(a) General case. Coupling method. C. (1986, 1989, 1990, 1994), C. Neuhauser (1990), Y. Li (1995).

Theorem 6.

- [C., 1990]: For polynomial model, $\beta_1, \cdots, \beta_{m_0}$ & $\delta_1, \cdots, \delta_{m_0+1}$ fixed, then ergodic $\forall \beta_0 \gg 1$.
- [C., 1994]: For Schlögl 2'nd model with $\beta_0 = 2\alpha$, $\beta_2 = 6\alpha$, $\delta_1 = 9\alpha$ and $\delta_3 = \alpha$, it is exponentially ergodic for all $\alpha \geqslant 0.7303$.

(b) Reversible case. Reaction part: birth rates $b(k)$ & death rates $a(k)$, RD-process reversible iff $p(u,v)\!=\!p(v,u)$ & $(k+1)b(k)/a(k)$=constant.

Theorem 7 [W. D. Ding, R. Durrett and T. M. Liggett, 1990; C., W. D. Ding & D. J. Zhu, 1994]. Reversible polynomial model always ergodic.

Replacing $\beta_0 > 0$ by $\beta_0 = 0$. Two stationary distr.

Theorem 8 [T. S. Mountford, 1990]. Reversible polynomial model with $\beta_0 = 0$, under mild assumption, Shiga's conjecture is correct.

8. Phase transitions.

(a) RD-processes with absorbing state.

(b) Mean field models.

(c) Linear growth model.

(a) RD-processes with absorbing state: $\beta_0 = 0$

Y. Li & X. G. Zheng (1988): color graph representation. R. Durrett (1988): oriented percolation.

Theorem 9. $S = \mathbf{Z}$, $b(k) = \lambda k$, $a(k) > 0\,(k \geqslant 1)$, diffusion rates $x_u p(u,v)$, $p(u,v)$ SRW. Then process $X^0(t)$ starting from x^0: $x_0^0 = 1$ & $x_u^0 = 0$, $\forall u \neq 0$, we have

$$\inf\{\lambda : X^0(t) \text{ never dies out}\} < \infty.$$

$\exists \lambda > 0$, $\exists$ non-trivial stationary distribution.

(b) Mean field models. Simpler & approximation. Easier to exhibit phase transitions.
Time-inhomogeneous birth-death: $a(k)$ usual, birth rates $b(k)+\mathbb{E}X(t)$. $X(t)$. $|\mathscr{I}|$: #{stationary distr.}

Theorem 10 [S. Feng & X. G. Zheng, 1992].
For 2'nd Schlögl model,

- $|\mathscr{I}| \geqslant 1$.
- $|\mathscr{I}| = 1$ if $\delta_1, \delta_3 \gg 1$.
- $|\mathscr{I}| \geqslant 3$ if $\delta_1 < \delta_1^2 < \frac{1}{2} + \frac{2\beta_2+1}{3\delta_1+6\delta_3}$ & $\beta_0 \ll 1$.

D. A. Dawson, S. Feng, X. G. Zheng, 5 papers.

Large deviations. J. W. Chen.

(c) Linear growth model.

W. D. Ding & Zheng (1989).

Problem 2. For polynomial model with $\beta_0 > 0$, $|\mathscr{I}| > 1$?

Meaning of $\beta_0 > 0$. Revised models:

R. Durrett et al, biology.

9. Hydrodynamic limits. Polynomial model. $\Omega^\varepsilon = \varepsilon^{-2}\Omega_d + \Omega_r$. $\mu^\varepsilon(\varepsilon > 0)$: product measure $\mu^\varepsilon(x_u) = \rho(\varepsilon u)$, $u \in \mathbf{Z}^d$, $0 \leqslant \rho \in C_b^2(\mathbf{R}^d)$. $\mathbb{E}_{\mu^\varepsilon}^\varepsilon$. C. Boldrighini, A. DeMasi, A. Pellegrinotti & E. Presutti (1987).

Theorem 11. $\forall r = (r^1, \cdots, r^d) \in \mathbf{R}^d$ & $t \geqslant 0$, limit $f(t,r) := \lim_{\varepsilon \to 0} \mathbb{E}_{\mu^\varepsilon}^\varepsilon X_{[r/\varepsilon]}(t)$, where $[r/\varepsilon] = ([r^1/\varepsilon], \cdots, [r^d/\varepsilon]) \in \mathbf{Z}^d$, exists and satisfies the RD-equation:

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{1}{2}\displaystyle\sum_{i=1}^{d} \frac{\partial^2 f}{\partial (r^i)^2} + \sum_{j=0}^{m} \beta_j f^j - \sum_{j=1}^{m+1} \delta_j f^j \\ f(0,r) = \rho(r). \end{cases} \tag{1}$$

Relation: RD-process & RD-equation.

C., L. P. Huang and X. J. Xu (1991); J. F. Feng (1996); T. Funaki (1997, 1999); A. Perrut (2000), $\cdots\cdots$

Let $\lambda \geqslant 0$ satisfy the algebraic equation:

$$\sum_{j=0}^{m} \beta_j \lambda^j - \sum_{j=1}^{m+1} \delta_j \lambda^j = 0. \qquad (2)$$

—Simplest solutions to 1'st of (1).

λ called **asymptotically stable** if $\exists \delta > 0$ s. t. for any solution $f(t, r)$ to (1), $|f(0, r) - \lambda| < \delta \Rightarrow \lim_{t\to\infty} |f(t, r) - \lambda| = 0$.

Theorem 12 [X. J. Xu, 1991]. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geqslant 0$ be roots of (2), λ_j have multiplicity m_j. Then, λ_i is asymptotically stable iff m_i is odd and $\sum_{j\leqslant i-1} m_j$ is even.

Conjecture: Model has no phase transition iff every λ_j is asymptotically stable.

Schlögl's 1'st model: Every λ_j is stable.

Schlögl's 2'nd: One is stable but not the other two.

Problem 3 (Conjecture).

(1) Schlögl's 1'st model has no phase transition.

(2) Schlögl's 2'nd model has phase transitions.

The forth mathematical tool.

Stochastic Model of Economic Optimization

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Stochastic Processes and Related Fields
Keio University, Japan
(December 9–12, 2002)[11]

Background

- May, 1989; S. Kusuoka. Financial Mathematics.
- The Chinese economy in that period.
- Theory of product of random matrices.

Contents

- Input–output method.
- L. K. Hua's fundamental theorem.
- Stochastic model without consumption.
- Stochastic model with consumption.

Part I. Input–output method

Fix the unit of the quantity of each product.

$$x = \left(x^{(1)}, x^{(2)}, \ldots, x^{(d)}\right)$$

vector of the quantities of the main products (*vector of products*).

Current economy. Examine three things.

- The vector of products input last year:

$$x_0 = (x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(d)}).$$

- The output of the vector of products this year:

$$x_1 = (x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(d)}).$$

- The *structure matrix* (or *matrix of expend coefficient*):

$$A_0 = (a_{ij}^{(0)}).$$

Meaning: to produce one unit of i-th product, one needs $a_{ij}^{(0)}$ units of the j-th product.

$$x_0^{(j)} = \sum_{i=1}^{d} x_1^{(i)} a_{ij}^{(0)}, \qquad x_0 = x_1 A_0.$$

Suppose all the products are used for the reproduction (*idealized model*). Then

$$x_{n-1} = x_n A_{n-1}, \qquad n \geqslant 1.$$

Hence

$$x_0 = x_1 A_0 = x_2 A_1 A_0 = x_n A_{n-1} \cdots A_0.$$

Time-homogeneous: $A_n = A, \quad n \geqslant 0.$
Simple expression for the n-th output:

$$x_n = x_0 A^{-n}, \qquad n \geqslant 1.$$

Known the structure matrix and the input x_0, predict the future output.
Well known *input-output method*.
In 1960's, more than 100 countries had used the method in their national economy.

Part Ⅱ. L. K. Hua's fundamental theorem

Return to the original equation $x_1 = x_0 A^{-1}$.
Fix A, then x_1 is determined by x_0 only.
Question: which choice of x_0 is the optimal one?
What sense?
Average of the ages of the members in a group.
Nursery.
Minimax principle: Find out x_0 such that $\min_{1\leqslant j\leqslant d} x_1^{(j)}/x_0^{(j)}$ attains the maximum below

$$\max_{x_1>0,\ x_0=x_1 A}\ \min_{1\leqslant j\leqslant d} x_1^{(j)}/x_0^{(j)}.$$

By using the classical Frobenius theorem,

Theorem [L. K. Hua, 1983]. Given an irreducible non-negative matrix A, let u be the left eigenvector (positive) of A corresponding to the largest eigenvalue $\rho(A)$ of A. Then, up to a constant, the solution to the above problem is $x_0 = u$. In this case, we have

$$x_1^{(j)} / x_0^{(j)} = \rho(A)^{-1} \quad \text{for all } j.$$

Called *the eigenvector method*.

Stability of economy. Expression:

$$x_n = x_0 \rho(A)^{-n} \quad \text{whenever } x_0 = u.$$

What happen if we take $x_0 \neq u$ (up to a constant)?

Set $T^x = \inf\{n \geqslant 1 : x_0 = x$ and there is some j such that $x_n^{(j)} \leqslant 0\}$

which is called the *collapse time* of the economic system.

Theorem [L. K. Hua, 1983]. Under some mild conditions, if $x_0 \neq u$, then $T^{x_0} < \infty$.

Example [Hua]. Consider two products only: industry and agriculture. Let

$$A = \frac{1}{100}\begin{pmatrix} 20 & 14 \\ 40 & 12 \end{pmatrix}.$$

Then $u = (5(\sqrt{2400} + 13)/7, 20)$. 44.34397483. We have

x_0	T^{x_0}
$(44, 20)$	3
$(44.344, 20)$	8
$(44.34397483, 20)$	13

Sensitive! Compare with Frobenius theorem or Brouwer fixed point theorem.

Interpretation

Particular case: $A = P$. A is a transition probability matrix. By ergodic theorem for MC,

$$P^n \to \mathbb{1}^* \pi \quad \text{as } n \to \infty,$$

where $\mathbb{1}$ is the row vector having elements 1, $\pi = \left(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)}\right)$ is stationary distribution of the corresponding MC. Since the distribution is the only stable solution for the chain, it should have some meaning in economics even though the later one goes in a converse way:

$$x_n = x_0 P^{-n}, \quad n \geqslant 1.$$

From the above facts, it is not difficult, as will be shown soon, to prove that if

$$x_0 \neq u = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d)})$$

up to a positive constant, then $T^{x_0} < \infty$.

Next, since the general case can be reduced to the above particular case, we think that this is a very natural way to understand the above Hua's theorem.

Proof. Need to show

$$x_n > 0 \text{ for all } n \Longrightarrow x_0 = \pi.$$

Let $x_0 > 0$ and $x_0 \mathbb{1}^* = 1$. Then

$$1 = x_0 \mathbb{1}^* = x_n P^n \mathbb{1}^* = x_n \mathbb{1}^*, \quad n \geqslant 1.$$

$\exists \{x_{n_k}\}_{k \geqslant 1}$ and $\bar{x}$: $\lim_{k\to\infty} x_{n_k} = \bar{x}$, $\bar{x} \geqslant 0$, $\bar{x}\mathbb{1}^* = 1$. Therefore,

$$x_0 = (x_0 P^{-n_k}) P^{n_k} = x_{n_k} P^{n_k} \to \bar{x} \mathbb{1}^* \pi = \pi.$$

We must have $x_0 = \pi$.

In L. K. Hua's eleven reports (1983–1985), he also studied some more general models of economy. The above two theorems are the key to his idea. The economy of markets was also treated. The only difference is that in the later case one needs to replace the structure matrix A with

$$V^{-1}AV,$$

where V is the diagonal matrix $\mathrm{diag}(v_i/p_i)$:
(p_i) is the vector of prices in a market,
(v_i) is the right eigenvector of A.

Note that the eigenvalues of $V^{-1}AV$ are the same as those of A. Corresponding to the largest eigenvalue

$$\rho(V^{-1}AV) = \rho(A),$$

the left eigenvector of $V^{-1}AV$ becomes

$$uV.$$

Thus, from mathematical point of view, the consideration of market makes no essential difference in the Hua's model.

Part Ⅲ. Stochastic model without consumption

Small random perturbation:

$$\widetilde{a}_{ij} = a_{ij} \quad \text{with probability } 2/3,$$
$$= a_{ij}(1 \pm 0.01) \quad \text{with probability } 1/6.$$

Taking $(\widetilde{a}_{ij})$ instead of (a_{ij}), we get a random matrix. Next, let $\{A_n;\, n \geqslant 1\}$ be a sequence of independent random matrices with the same distribution as above, then

$$x_n = x_0 \prod_{k=1}^{n} A_k^{-1}$$

gives us a stochastic model of an economy without consumption.

Starting from $x_0 = (44.344,\ 20)$, then the collapse probability in stochastic model is the following

$$\mathbb{P}[T^{x_0} = n] = \begin{cases} 0, & \text{for } n = 1, \\ 0.09, & \text{for } n = 2, \\ 0.65, & \text{for } n = 3. \end{cases}$$

$$\mathbb{P}[T \leqslant 3] \approx 0.74.$$

This observation tells us that randomness plays a critical role in the economy. It also explains the reason why the input-output is not very practicable, as people often think, because the randomness has been ignored and so the deterministic model is far away from the real practice.

What is the analog of Hua's theorem for the stochastic case?

Theorem [C., 1992]. Under some mild conditions, we have

$$\mathbb{P}[T^{x_0} < \infty] = 1, \qquad \forall x_0 > 0.$$

Proof is not easy! We have to deal with the product of random matrices:

$$M_n = A_n A_{n-1} \cdots A_1.$$

Liapynov exponent ("strong law of large numbers"). $\|A\|$: the operator norm of A.

Theorem. Let $\mathbb{E}\log^+\|A_1\| < \infty$. Then

$$\frac{1}{n}\log\|M_n\| \xrightarrow{\text{a.s.}} \gamma \in \{-\infty\} \cup \mathbb{R},$$

where $\gamma = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\log\|M_n\|$.

Theorem [H. Kesten & F. Spitzer: PTRF, 1984]. $M_n/\|M_n\|$ converges in distribution to a positive matrix $M = L^*R$ with rank one, where L and R are positive row vectors.

— Strong law of large numbers

Recent paper: Ann. of Probab. (1997), No. 4.

Part IV. Stochastic model with consumption

The model without consumption is idealized! More practical one should have consumption. Allow a part of the productions turning into consumption, not used for reproduction.

Suppose that every year we take the $\theta^{(i)}$-times amount of the increment of the i-th product to be consumed. Then in the first year, the vector of products which can be used for reproduction is

$$y_1 = x_0 + (x_1 - x_0)(I - \Theta),$$

where I is the $d \times d$ unit matrix and $\Theta = \operatorname{diag}\left(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(d)}\right)$, which is called a *consumption matrix*. Therefore,

$$y_1 = y_0[A_0^{-1}(I - \Theta) + \Theta], \qquad y_0 = x_0.$$

Similarly, in the n-th year, the vector of the products which can be used for reproduction is

$$y_n = y_0 \prod_{k=0}^{n-1} [A_{n-k-1}^{-1}(I - \Theta) + \Theta].$$

Let
$$B_n = [A_{n-1}^{-1}(I - \Theta) + \Theta]^{-1}.$$

Then
$$y_n = y_0 \prod_{k=1}^{n} B_{n-k+1}^{-1}.$$

We have thus obtained a *stochastic model with consumption*. In the deterministic case, a collapse theorem was obtained by L. K. Hua and S. Hua (1985). More stable! Dim $(x_0) > 1$.

- $Gl(d,\mathbb{R})$: General linear group of real invertible $d \times d$ matrices.
- $O(d,\mathbb{R})$: Orthogonal matrices in $Gl(d,\mathbb{R})$.
- $\mathscr{G}_\mu$: Smallest closed semigroup of $Gl(d,\mathbb{R})$ containing $\operatorname{supp}(\mu)$.
- $\mathscr{G}$ strongly irreducible: $\nexists$ proper linear subspaces of $\mathbb{R}^d$, $\mathscr{V}_1, \cdots, \mathscr{V}_k$ such that $(\cup_{i=1}^k \mathscr{V}_i)B = \cup_{i=1}^k \mathscr{V}_i, \forall B \in \mathscr{G}$.
- $\mathscr{G}$ contractive: $\exists \{B_n\} \subset \mathscr{G}$ such that $\|B_n\|^{-1} B_n$ converges to a matrix with rank one.
- Polar decomposition: $B = K \operatorname{diag}(a_i) U$, $K, U \in O(d,\mathbb{R}),\ a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d > 0$.

Theorem [C., Y. Li, 1994]. Let $\{B_n\}$ i.i.d. $\sim \mu$. Suppose that $\mathscr{G}_\mu$ strongly irreducible, contractive and "K" satisfies a tightness condition. Then

$$\mathbb{P}[T^x < \infty] = 1 \text{ for all } 0 < x \in \mathbb{R}^d.$$

Open problems:

- How fast does the economy go to collapse?

$$\mathbb{P}[T > n] \leqslant Ce^{-\alpha n}.$$

- How to control the economy and what is the optimal one?

http://math.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

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